Midterm I
Math 5B, UCSB, Spring '06

You have 50 minutes to complete this exam.

Name:____________________

Perm #:____________________

Signature:____________________

Discussion section:____________________

Show all your work. Partial credit will be given only if work is relevant and correct. *Please make your work as clear and easy to follow as possible.* You might want to put scratch work on the back of every sheet, and put neat clean work on the front of every sheet. You don't need to simplify your answers but you need to justify them.

Problem	Possible Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Extra	10	
Total	50 (+10)	

Name:____________________________

Exercise 1.

Determine if the following limits exist. If so, determine their value (Justify your answer):

1) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2} =$ does not exist

For example, if we have $y=0$ and let x tend to 0 from the positive direction we get

$$\frac{x-y}{x^2+y^2} = \frac{x}{x^2} = \frac{1}{x} \longrightarrow +\infty$$

however, if x tends to 0 from the negative direction we get $\frac{x-y}{x^2+y^2} = \frac{x}{x^2} = \frac{1}{x} \longrightarrow -\infty$

2) $\lim_{(x,y)\to(0,0)} \frac{x^3+x^2+y^2}{x^2+y^2} =$

This limit equals 1

Convert to polar coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+x^2+y^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^2}{r^2}$$

$$= \lim_{r\to 0} r\cos^3\theta + 1 = 1$$

Name:________________________________

Exercise 2.

Find the parametric representation of a circle in $\mathbb{R}^3$ with radius $= 1$, centered at the point $(1, 1, 1)$ and contained in the plane $z = 1$.

One such solution is

$$x = \cos\theta + 1$$
$$y = \sin\theta + 1$$
$$z = 1$$

for $0 \leq \theta \leq 2\pi$

Name:______________________________

Exercise 3.

Find the tangent plane of the surface in $\mathbb{R}^3$:

$$x^2 + y^2 - z^2 = 1$$

at the point $(1, 1, 1)$.

Since the surface is a level set of the function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(x, y, z) \longmapsto x^2 + y^2 - z^2$$

then $\nabla f(1,1,1) = (2, 2, -2)$ is a vector normal to the surface at $(1,1,1)$.

Therefore, the tangent plane we seek can be described as the set of points $(x,y,z) \in \mathbb{R}^3$ satisfying

$$(x-1, y-1, z-1) \cdot (2, 2, -2) = 0$$

In other words, the plane is

$$2(x-1) + 2(y-1) - 2(z-1) = 0$$

Name:________________________

Exercise 4.

Show that the curve in $\mathbb{R}^3$, described by

$$\begin{cases} x(t) = t\cos(t) \\ y(t) = t\sin(t) \\ z(t) = t \end{cases}$$

is contained inside the surface (cone in $\mathbb{R}^3$):

$$x^2 + y^2 = z^2$$

and compute its tangent line at the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

The curve described above is in the cone because

$$(t\cos t)^2 + (t\sin t)^2 = t^2$$

for all t.

The tangent line at the point $p = (0, \pi/2, \pi/2)$

Let $c(t) = (t\cos t, t\sin t, t)$.

We have $c(\pi/2) = p$. So the curve passes through p at $t = \pi/2$. Thus the tangent line is parametrized by

$$(x, y, z) = p + c'(\pi/2)t$$

Since $c'(t) = (-t\sin t + \cos t, t\cos t + \sin t, 1)$

$$c'(\pi/2) = (-\pi/2, 1, 1)$$

The tangent line is $\boxed{(x, y, z) = (-\pi/2\, t, \pi/2 + t, \pi/2 + t)}$

Name:____________________

Exercise 5.

Given the transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$:

$$F = \begin{cases} z_1 = y_1 y_2 - 3y_1 \\ z_2 = y_1 - y_2^2 \end{cases}$$

$$G = \begin{cases} y_1 = x_1 \\ y_2 = x_1 + x_2 \end{cases}$$

Compute the Jacobian matrices of F, G and their composition $H = FG$.

$$\text{Jac } F = \begin{bmatrix} \frac{\partial z_1}{\partial y_1} & \frac{\partial z_1}{\partial y_2} \\ \frac{\partial z_2}{\partial y_1} & \frac{\partial z_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} y_2 - 3 & y_1 \\ 1 & -2y_2 \end{bmatrix}$$

$$\text{Jac } G = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{Jac } H = (\text{Jac } F)(\text{Jac } G) = \begin{bmatrix} y_2 - 3 & y_1 \\ 1 & -2y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 + y_2 - 3 & y_1 \\ 1 - 2y_2 & -2y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 + x_2 - 3 & x_1 \\ 1 - 2(x_1 + x_2) & -2(x_1 + x_2 \end{bmatrix}$$

Exercise 6. (Extra)

Show that the curve in $\mathbb{R}^3$ given by the equations

$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x - y = 0 \end{cases}$$

is orthogonal to the plane $x + y = 0$ at the point $(0, 0, 1)$ (i,e, the tangent line to the curve at that point is orthogonal to the plane).

This curve can be parametrized by

$$x(t) = \tfrac{\sqrt{2}}{2} \sin t$$
$$y(t) = \tfrac{\sqrt{2}}{2} \sin t$$
$$z(t) = \cos t$$

Under this parametrization, the curve passes through $(0,0,1)$ when $t=0$.

The tangent line to the curve at $(0,0,1)$ points in the direction

$$\vec{v} = (x'(0), y'(0), z'(0)) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

The plane P given by $x+y=0$ can be described as the set of points $(x,y,z) \in \mathbb{R}^3$ satisfying $(1,1,0) \cdot (x,y,z) = 0$. Thus, if (x,y,z) is in the plane P, then

$$\vec{v} \cdot (x,y,z) = \tfrac{\sqrt{2}}{2} (1,1,0) \cdot (x,y,z) = 0$$

Therefore, the curve is orthogonal to P at $(0,0,1)$.