Math 5B Midterm Exam
Spring 2006

Your name:
Your perm.:
Your signature:

Scores:
1.
2.
3.
4.
5.
Total: (out of 100)

Note: 12 extra credit points are included.

1. (22 points) 1)(15 points) Find the differential dw of the function

$$w = f(x, y, z) = x^2yz + xy^3e^z.$$

2)(4 points) What is the value of this differential at $x_0 = 1, y_0 = 1, z_0 = 0$ when $\Delta x = 0.1, \Delta y = 0.1, \Delta z = 0.2$?
3) (3 points) Find an approximate value for $f(1.1, 1.1, 0.2)$.

Solution 1) We have

$$f_x = 2xyz + y^3e^z, f_y = x^2z + 3xy^2e^z, f_z = x^2y + xy^3e^z.$$

Hence

$$dw = f_xdx+f_ydy+f_zdz = (2xyz+y^3e^z)dx+(x^2z+3xy^2e^z)dy+(x^2y+xy^3e^z)dz.$$

2) At $x_0 = 1, y_0 = 1, z_0 = 0$ we have by the above result

$$dw = (2\cdot1\cdot1\cdot0+1^3\cdot e^0)dx+(1^2\cdot0+3\cdot1\cdot1^2\cdot e^0)dy+(1^2\cdot1+1\cdot1^3\cdot e^0)dz = dx+3dy+2dz.$$

When $dx = \Delta x = 0.1, dy = \Delta y = 0.1$ and $dz = \Delta z = 0.2$ we obtain

$$dw = 0.1 + 0.3 + 0.4 = 0.8.$$

3) We have

$$f(1.1, 1.1, 0.2) = f(1, 1, 0) + \Delta w \approx f(1, 1, 0) + dw,$$

where $\Delta w = f(1.1, 1.1, 0.2) - f(1, 1, 0)$ and dw is at $(1, 1, 0)$ for $dx = 0.1, dy = 0.1, dz = 0.2$. We have $f(1, 1, 0) = 1$. By 2) we then deduce

$$f(1.1, 1.1, 0.2) \approx 1 + 0.8 = 1.8.$$

2. (30 points) **Do any two. Do not do three. Only the first two problems will be graded if you do three.**

1) (15 points) Find the equation of the plane which contains the points $A = (3, 1, 1), B = (1, 2, 3)$ and $C = (2, 0, 2)$.

2)(15 points) Find the equation of the tangent plane for the level surface $x^2y + y^2z + z^2x = 3$ at $(1, 1, 1)$.

3) (15 points) Find the equation of the tangent plane of the graph of $z = x^2 + 3y^2$ at $x = 1, y = 1$.

Solution 1) Let $\mathbf{v}$ denote the vector from A to B, and $\mathbf{w}$ the vector from A to C. Then

$$\mathbf{v} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{w} = -\mathbf{i} - \mathbf{j} + \mathbf{k}.$$

We have

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ -1 & -1 & 1 \end{pmatrix} = 3\mathbf{i} + 3\mathbf{k}.$$

It follows that the desired equation is

$$3(x - 3) + 3(z - 1) = 0, \text{ i.e. } x + z = 4.$$

2) Set $f(x, y, z) = x^2y + y^2z + z^2x$. Then

$$\nabla f = (2xy + z^2, x^2 + 2yz, y^2 + 2zx)$$

and

$$\nabla f(1, 1, 1) = (3, 3, 3).$$

It follows that the desired equation is

$$3(x - 1) + 3(y - 1) + 3(z - 1) = 0, \text{ i.e. } x + y + z = 1.$$

3) We have $z_x = 2x, z_y = 6y$ and hence $z_x(1, 1) = 2, z_y(1, 1) = 6$. We also have $z(1, 1) = 1 + 3 = 4$. It follows that the desired equation is

$$z = 4 + 2(x - 1) + 6(y - 1) = 2x + 6y - 4.$$

3. (18 points) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where x, y and z satisfy $\ln(xy+1)+\ln(yz+1)+\ln(xy+1)=1$.

Solution *Method 1* We set $F(x,y,z)=\ln(xy+1)+\ln(yz+1)+\ln(xy+1)-1=2\ln(xy+1)+\ln(yz+1)-1$. Then

$$F_x=\frac{2y}{xy+1}, F_y=\frac{2x}{xy+1}+\frac{z}{yz+1}, F_z=\frac{y}{yz+1}.$$

Then we have

$$z_x=-\frac{F_x}{F_z}=-\frac{2(yz+1)}{xy+1}, z_y=-\frac{F_y}{F_z}=-\frac{2x(yz+1)}{y(xy+1)}-\frac{z}{y}.$$

Method 2 We use differentials. Taking differentials of the given equation we obtain

$$2\frac{xdy+ydx}{xy+1}+\frac{zdy+ydz}{yz+1}=0.$$

It follows that

$$2\frac{y}{xy+1}dx+(2\frac{x}{xy+1}+\frac{z}{yz+1})dy+\frac{y}{yz+1}dz=0.$$

Hence

$$dz=-2\frac{yz+1}{xy+1}dx-(\frac{2x(yz+1)}{y(xy+1)}+\frac{z}{y})dy.$$

Since $dz=z_xdx+z_ydy$ we infer the same formulas for z_x and z_y as above.

The corrected version:

$$\ln(xy+1)+\ln(yz+1)+\ln(xz+1)=1.$$

Solution Set $F(x,y,z)=\ln(xy+1)+\ln(yz+1)+\ln(xz+1)-1$. Then

$$F_x=\frac{y}{xy+1}+\frac{z}{xz+1}, F_y=\frac{x}{xy+1}+\frac{z}{yz+1}, F_z=\frac{y}{yz+1}+\frac{x}{xz+1}.$$

Then

$$z_x=-\frac{F_x}{F_z}=-\frac{\frac{y}{xy+1}+\frac{z}{xz+1}}{\frac{y}{yz+1}+\frac{x}{xz+1}}, z_y=-\frac{F_y}{F_z}=-\frac{\frac{x}{xy+1}+\frac{z}{yz+1}}{\frac{y}{yz+1}+\frac{x}{xz+1}}.$$

(One can also use Method 2.)

4. (20 points) Consider the function $F(u,v) = (e^{uv}, e^{2u-3v})$. Let (u,v) be given by a function $G(x,y) = (g_1(x,y), g_2(x,y))$, such that

$$\frac{\partial g_1}{\partial x} = xy^2, \frac{\partial g_1}{\partial y} = x^2 y$$

and

$$\frac{\partial g_2}{\partial x} = x + y, \frac{\partial g_2}{\partial y} = x.$$

1) (17 points) Use the chain rule to find the Jacobian matrix of the composite function $F \circ G$, i.e. $F(g_1(x,y), g_2(x,y))$.
2) (3points) Find the Jacobian determinant of $F \circ G$.

Solution 1) We have

$$DF = \begin{pmatrix} \frac{\partial}{\partial u} e^{uv} & \frac{\partial}{\partial v} e^{uv} \\ \frac{\partial}{\partial u} e^{2u-3v} & \frac{\partial}{\partial v} e^{2u-3v} \end{pmatrix} = \begin{pmatrix} ve^{uv} & ue^{uv} \\ 2e^{2u-3v} & -3e^{2u-3v} \end{pmatrix}.$$

By the chain rule we then have

$$D(F \circ G) = DF \cdot DG = \begin{pmatrix} ve^{uv} & ue^{uv} \\ 2e^{2u-3v} & -3e^{2u-3v} \end{pmatrix} \cdot \begin{pmatrix} xy^2 & x^2 y \\ x + y & x \end{pmatrix}$$

$$= \begin{pmatrix} (xy^2 v + (x+y)u)e^{uv} & x(xyv + u)e^{uv} \\ (2xy^2 - 3(x+y))e^{2u-3v} & x(2xy - 3)e^{2u-3v} \end{pmatrix},$$

where $u = g_1(x,y)$ and $v = g_2(x,y)$. (Note: It is not hard to show that $g_1(x,y) = \frac{1}{2}x^2y^2 + C_1$ and $g_2(x,y) = \frac{1}{2}x^2 + xy + C_2$ for some constants C_1 and C_2.)

2) We have

$$J_{F \circ G} = J_F \cdot J_G = -(3v + 2u)e^{uv+2u-3v} \cdot (-x^3 y) = x^3 y(2u + 3v)e^{uv+2u-3v},$$

where $u = g_1(x,y)$ and $v = g_2(x,y)$.

5. (22 points) 1)(20 points) Find all the second order partial derivatives of the function

$$z = f(x, y) = \ln(x^2 + y^2).$$

2) (2 points) Show that f satisfies the Laplace equation $\Delta f = 0$.

Solution 1) We have

$$z_x = \frac{2x}{x^2 + y^2}, z_y = \frac{2y}{x^2 + y^2}.$$

Hence

$$z_{xx} = \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2},$$

$$z_{xy} = z_{yx} = -\frac{4xy}{(x^2 + y^2)^2},$$

$$z_{yy} = \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}.$$

2) We have

$$\Delta z = z_{xx} + z_{yy} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0.$$