Name: Solutions

Section Time :

Math 5B - Final Exam

December 15, 2006

Instructions:

- This exam consists of 11 problems worth 10 points each, but will be graded out of 100 points.
- You must show all your work and fully justify your answers in order to recieve full credit. You may leave your answers in unsimplified form, unless the problem asks you to simplify.
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.
- Some useful trig identities include:

$$\sin^2 x + \cos^2 x = 1 \qquad \sin^2 x = \tfrac{1}{2}(1-\cos(2x))$$

$$\sin(2x) = 2\sin x\cos x \qquad \cos^2 x = \tfrac{1}{2}(1+\cos(2x))$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

1		7	
2		8	
3		9	
4		10	
5		11	
6		Total	

1. (a) Find parametric equations for the line that is perpendicular to the two lines $l_1 : (x, y, z) = (2t, -t, 0)$ and $l_2 : (x, y, z) = (-3t, 1, t+2)$ and passes through the point $(1, 1, 1)$.

l_1 has direction vector $(2,-1,0)$
l_2 has direction vector $(-3,0,1)$

The direction vector we want is

$$\vec{v} = (2,-1,0) \times (-3,0,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 0 \\ -3 & 0 & +1 \end{vmatrix} = -\vec{i} - 2\vec{j} - 3\vec{k} = (-1,-2,-3)$$

So $\boxed{\begin{aligned} x &= 1-t \\ y &= 1-2t \\ z &= 1-3t \end{aligned}}$

(b) Find the equation of the tangent plane to the surface $x^2 + xy + 4y\sqrt{z} = 2$ at the point $(3, -1, 1)$.

Let $F(x,y,z) = x^2 + xy + 4y\sqrt{z} - 2$

The normal vector is $\nabla F\big|_{(3,-1,1)} = \left(2x+y,\, x+4\sqrt{z},\, 2y/\sqrt{z}\right)\Big|_{(3,-1,1)}$

$\Rightarrow \vec{n} = (5,7,-2)$

So the tangent plane has equation

$$\boxed{5(x-3) + 7(y+1) - 2(z-1) = 0}$$

2. Suppose $x(u,v) = u + 2v + u^3$ and $y(u,v) = uv - v^3$.

(a) Find the Jacobian matrix $\frac{\partial(u,v)}{\partial(x,y)}$ of the inverse mapping when $u = 1$ and $v = 1$.

$x(1,1) = 4, \quad y(1,1) = 0$

$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} = \begin{pmatrix} 1+3u^2 & 2 \\ v & u-3v^2 \end{pmatrix}^{-1}$$

at $u=1, v=1$
$$= \begin{pmatrix} 4 & 2 \\ 1 & -2 \end{pmatrix}^{-1} = \frac{1}{-10}\begin{pmatrix} -2 & -2 \\ -1 & 4 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} .2 & .2 \\ .1 & -.4 \end{pmatrix}}$$

(b) Use part (a) to approximate the values of u and v when $x = 3.9$ and $y = 0.1$.

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)\begin{pmatrix} dx \\ dy \end{pmatrix}$$

Since $x(1,1) = 4$ and $y(1,1) = 0$, $dx = -0.1$ & $dy = 0.1$

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} .2 & .2 \\ .1 & -.4 \end{pmatrix}\begin{pmatrix} -.1 \\ .1 \end{pmatrix} = \begin{pmatrix} -.02 + .02 \\ -.01 - .04 \end{pmatrix} = \begin{pmatrix} 0 \\ -.05 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -.05 \end{pmatrix} = \begin{pmatrix} 1 \\ .95 \end{pmatrix}$$

So $\boxed{u \approx 1 \text{ and } v \approx .95}$

3. Let $f(x,y) = 9x^2y - 6x^3 + y^3 - 3y$. Find all critical points of $f(x,y)$ and classify each as a relative maximum, relative minimum or saddle point.

$$\nabla f = (18xy - 18x^2,\ 9x^2 + 3y^2 - 3) = (0,0)$$

$$18xy - 18x^2 = 18x(y-x) = 0$$

$$\Rightarrow x = 0 \text{ or } y = x.$$

<u>if $x = 0$:</u>

$$9x^2 + 3y^2 - 3 = 0$$

$$\Rightarrow 3y^2 - 3 = 0$$

$$3y^2 = 3$$

$$y = \pm 1$$

<u>if $y = x$</u>

$$9x^2 + 3y^2 - 3 = 0$$

$$\Rightarrow 12x^2 = 3$$

$$x^2 = 1/4$$

$$x = \pm \tfrac{1}{2}$$

<u>Critical points</u>

$$\boxed{(0,1),\ (0,-1),\ \left(\tfrac{1}{2},\tfrac{1}{2}\right),\ \left(-\tfrac{1}{2},-\tfrac{1}{2}\right).}$$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial^2 y} \end{bmatrix} = \begin{bmatrix} 18y - 36x & 18x \\ 18x & 6y \end{bmatrix}$$

- $H(0,1) = \begin{bmatrix} 18 & 0 \\ 0 & 6 \end{bmatrix}$ $\left.\begin{array}{l} \det(H) = 18 \cdot 6 > 0 \\ tr(H) = 18 + 6 > 0 \end{array}\right\} \Rightarrow \boxed{\text{Rel. Min @ } (0,1)}$

- $H(0,-1) = \begin{bmatrix} -18 & 0 \\ 0 & -6 \end{bmatrix}$ $\left.\begin{array}{l} \det(H) = (-18)(-6) > 0 \\ tr(H) = -18 - 6 < 0 \end{array}\right\} \Rightarrow \boxed{\text{Rel. Max @ } (0,-1)}$

- $H\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \begin{bmatrix} -9 & 9 \\ 9 & 3 \end{bmatrix}$ $\det(H) = -27 - 81 < 0 \Rightarrow \boxed{\text{SADDLE PT. @ } \left(\tfrac{1}{2},\tfrac{1}{2}\right)}$

- $H\left(-\tfrac{1}{2},-\tfrac{1}{2}\right) = \begin{bmatrix} 9 & -9 \\ -9 & -3 \end{bmatrix}$ $\det(H) = -27 - 81 < 0$ $\Rightarrow \boxed{\text{SADDLE PT. @ } \left(-\tfrac{1}{2},-\tfrac{1}{2}\right)}$

4. Consider the vector field $\mathbf{v} = (\sin x \sin y)\mathbf{i} + (z - \cos y)\mathbf{j} + (\sin z \cos x)\mathbf{k}$ on $\mathbb{R}^3$.

(a) Find div($\mathbf{v}$).

$$\mathrm{div}(\vec{v}) = \frac{\partial}{\partial x}(\sin x \sin y) + \frac{\partial}{\partial y}(z - \cos y) + \frac{\partial}{\partial z}(\sin z \cos x)$$

$$= \boxed{\sin y \cos x + \sin y + \cos x \cos z}$$

(b) Find curl($\mathbf{v}$).

$$\mathrm{curl}(\vec{v}) = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x \sin y & z - \cos y & \sin z \cos x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(\sin z \cos x) - \frac{\partial}{\partial z}(z - \cos y)\right)\vec{i} + \left(\frac{\partial}{\partial z}(\sin x \sin y) - \frac{\partial}{\partial x}(\sin z \cos x)\right)\vec{j}$$

$$+ \left(\frac{\partial}{\partial x}(z - \cos y) - \frac{\partial}{\partial y}(\sin x \sin y)\right)\vec{k}$$

$$= \boxed{-\vec{i} + (\sin x \sin z)\vec{j} - (\sin x \cos y)\vec{k}}$$

(c) Find curl(∇(div($\mathbf{v}$))).

$\mathrm{curl}(\nabla(\mathrm{div}(\vec{v}))) = \boxed{0}$ since the curl of a gradient is always 0.

5. Integrate

(a) $\int\int_R (x^2 - y^2)\, dx\, dy$ where R is the circle of radius 2, centered at the origin.

Convert to Polar Coordinates: $x = r\cos\theta$, $y = r\sin\theta$

R: $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$$\iint_R (x^2-y^2)\,dx\,dy = \int_0^{2\pi}\int_0^2 (r^2\cos^2\theta - r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3(\cos^2\theta - \sin^2\theta)\, dr\, d\theta$$

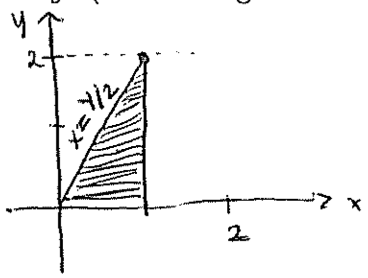
$$= \int_0^{2\pi} \cos(2\theta)\, \frac{r^4}{4}\Big]_0^2\, d\theta = \int_0^{2\pi} 4\cos(2\theta)\, d\theta = 2\sin(2\theta)\Big]_0^{2\pi}$$

$$= \boxed{0}$$

(b) $\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy$. (Hint: integrate with respect to y first.)

$0 \le y \le 2$

$\frac{y}{2} \le x \le 1$



$0 \le x \le 1$

$0 \le y \le 2x$

$$\int_0^2 \int_{y/2}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx$$

$$= \int_0^1 y e^{x^2}\Big]_0^{2x}\, dx$$

$$= \int_0^1 2x e^{x^2}\, dx = \int_0^1 e^u\, du$$

let $u = x^2$

$du = 2x\, dx$

$$= e^u\Big]_0^1 = \boxed{e-1}$$

6. Consider the surface S defined by $z = xy$ above the region

$$R = \{(x, y) \mid x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0\},$$

i.e., R is the first quadrant of the unit disk in the xy-plane.

(a) Find the surface area of S.

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

$$= \iint_R \sqrt{1 + y^2 + x^2}\, dx\, dy$$

Convert to Polar Coordinates: $R: 0 \leq \theta \leq \pi/2,\ 0 \leq r \leq 1$

$$= \int_0^{\pi/2} \int_0^1 \sqrt{1 + r^2}\, r\, dr\, d\theta = \int_0^{\pi/2} \int_1^2 \frac{1}{2}\sqrt{u}\, du\, d\theta$$

u-Substitution: let $u = 1 + r^2$, $du = 2r\, dr$

$$= \int_0^{\pi/2} \frac{1}{3} u^{3/2}\Big]_1^2 d\theta = \int_0^{\pi/2} \frac{1}{3}(2\sqrt{2} - 1)\, d\theta$$

$$= \boxed{\frac{\pi}{6}(2\sqrt{2} - 1)}$$

(b) Find the volume of the region under S and above R.

$$V = \iint_R z\, dx\, dy = \iint_R xy\, dx\, dy$$

Convert to Polar Coordinates:

$$= \int_0^{\pi/2} \int_0^1 r^2 \cos\theta \sin\theta\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta \sin\theta\, \frac{r^4}{4}\Big]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\sin(2\theta)\left(\frac{1}{4}\right) d\theta$$

$$= \frac{1}{8}\left(-\frac{\cos(2\theta)}{2}\right)\Big]_0^{\pi/2} = \frac{1}{8}\left(\frac{1}{2} - \frac{-1}{2}\right) = \boxed{\frac{1}{8}}$$

7. Let C be the curve $y = 9 - x^2$ from $(0,9)$ to $(3,0)$. Evaluate

(a) $\int_C xy\,dx - x^2\,dy$.

CHECK FOR PATH-INDEPENDENCE:

$$\frac{\partial}{\partial y}(xy) = x \neq \frac{\partial}{\partial x}(-x^2) = -2x. \quad \text{NO.}$$

Parameterize C: $\begin{cases} x = x \\ y = 9 - x^2 \\ 0 \le x \le 3 \end{cases}$ $\quad dx = dx$, $\quad dy = -2x\,dx$

$$\int_C xy\,dx - x^2\,dy = \int_0^3 [x(9-x^2) - x^2(-2x)]\,dx$$

$$= \int_0^3 9x + x^3\,dx = \left(\frac{9x^2}{2} + \frac{x^4}{4}\right)\Big]_0^3 = \frac{81}{2} + \frac{81}{4}$$

$$= \boxed{\frac{243}{4}}$$

(b) $\int_C (2xy^4 + y^3 - 1)\,dx + (4x^2y^3 + 3xy^2 + 1)\,dy$.

Check for Path-Independence:

$$\left.\begin{aligned} \frac{\partial}{\partial y}(2xy^4 + y^3 - 1) &= 8xy^3 + 3y^2 \\ \frac{\partial}{\partial x}(4x^2y^3 + 3xy^2 + 1) &= 8xy^3 + 3y^2 \end{aligned}\right\} =$$

so the integral is path-independent.

Look for $F(x,y)$ with $\frac{\partial F}{\partial x} = 2xy^4 + y^3 - 1$

and $\frac{\partial F}{\partial y} = 4x^2y^3 + 3xy^2 + 1$

$$F = \int 2xy^4 + y^3 - 1\,dx = x^2y^4 + xy^3 - x + C(y)$$

$$\frac{\partial F}{\partial y} = 4x^2y^3 + 3xy^2 + C'(y) = 4x^2y^3 + 3xy^2 + 1$$

$$\Rightarrow C'(y) = 1 \Rightarrow C(y) = y.$$

so $F(x,y) = x^2y^4 + xy^3 - x + y$.

$$\int_{C\,(0,9)}^{(3,0)} (2xy^4 + y^3 - 1)\,dx + (4x^2y^3 + 3xy^2 + 1)\,dy$$

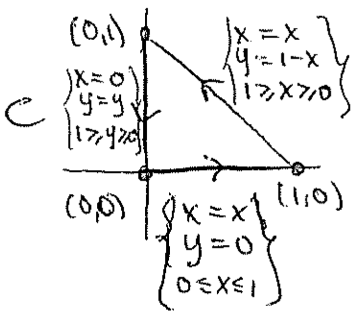
$$= F(3,0) - F(0,9) = -3 - 9$$

$$= \boxed{-12}$$

8. Let C be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ traversed in the counter-clockwise direction. Calculate:

(a) $\oint_C xy\, ds$.

$$\oint_C xy\,ds = \int_{(0,0)}^{(1,0)} xy\,ds + \int_{(1,0)}^{(0,1)} xy\,ds + \int_{(0,1)}^{(0,0)} xy\,ds$$



$$= \int_0^1 x\cdot 0\sqrt{1^2+0^2}\,dx + \int_1^0 x(1-x)\sqrt{1^2+(-1)^2}\,dx$$

$$+ \int_1^0 0\cdot y\sqrt{0^2+1^2}\,dy$$

$$= 0 + \int_1^0 (x-x^2)\sqrt{2}\,dx + 0$$

$$= \sqrt{2}\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big]_1^0 = -\sqrt{2}\left(\frac{1}{2}-\frac{1}{3}\right) = \boxed{-\frac{\sqrt{2}}{6}}$$

(b) $\oint_C [\sin(x+y) + x\cos(x+y)]\, dx + [y + x\cos(x+y)]\, dy$.

By Green's Theorem

$$= \iint_R \left[\frac{\partial}{\partial x}\left(y + x\cos(x+y)\right) - \frac{\partial}{\partial y}\left(\sin(x+y) + x\cos(x+g)\right)\right] dx\,dy$$

$$= \iint_R \left(-x\sin(x+y) + \cos(x+y) - \cos(x+y) + x\sin(x+y)\right) dx\,dy$$

$$= \iint_R 0\,dx\,dy = \boxed{0}$$

9. Integrate $\oint_C [e^{xy} + xye^{xy}]\, dx + [x^2e^{xy} + x]\, dy$ where C is the square $|x| + |y| = 1$ traversed counter-clockwise.

By Green's Theorem

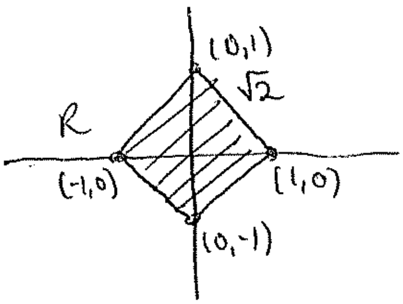
$$\oint_C (e^{xy} + xye^{xy})\,dx + (x^2e^{xy} + x)\,dy$$

$$= \iint_R \frac{\partial}{\partial x}(x^2e^{xy} + x) - \frac{\partial}{\partial y}(e^{xy} + xye^{xy})\ dx\,dy$$

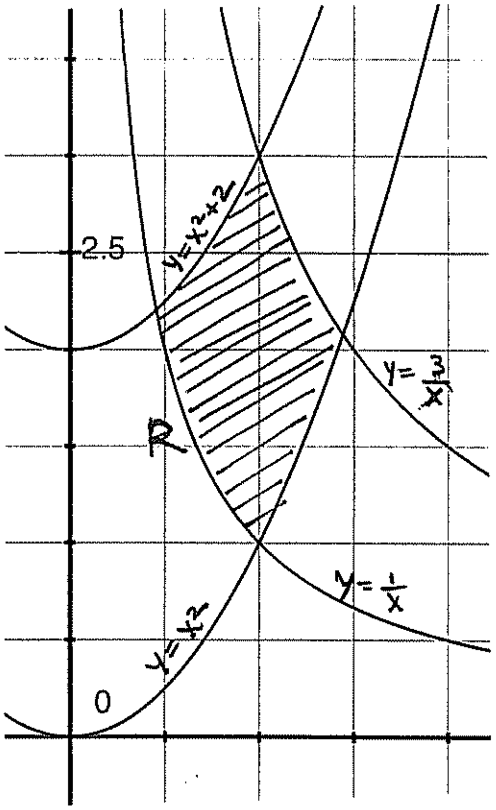
$$= \iint_R (2xe^{xy} + x^2ye^{xy} + 1 - xe^{xy} - xe^{xy} - x^2ye^{xy})\,dx\,dy$$

$$= \iint_R 1\,dx\,dy = \text{Area of } R = (\sqrt{2})^2 = \boxed{2}$$

R = Square of side length $= \sqrt{2}$.



10. Let R be the region in the first quadrant bounded by the curves $y = x^2$, $y = x^2+2$, $y = 1/x$ and $y = 3/x$. Use the change of coordinates $u = y - x^2$, $v = xy$ to integrate



$$\int\int_R \left(\frac{x}{y} - \frac{1}{x}\right)(4x^2+2y)\,dx\,dy.$$

R is given by the Inequalities:

$$2 \geq y - x^2 \geq 0 \implies 2 \geq u \geq 0$$

and $1 \leq xy \leq 3 \implies 1 \leq v \leq 3.$

The Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} -2x & 1 \\ y & x \end{vmatrix}^{-1}$$

$$= \left|(-2x^2 - y)^{-1}\right|$$

$$= \frac{1}{2x^2+y}$$

$$\therefore \iint_R \left(\frac{x}{y} - \frac{1}{x}\right)(4x^2+2y)\,dx\,dy = \int_1^3\int_2^0 \left(\frac{x^2-y}{xy}\right)\frac{4x^2+2y}{2x^2+y}\,du\,dv$$

$$= \int_1^3\int_2^0 \frac{-u}{v}\cdot 2\,du\,dv$$

$$= \int_1^3 \left.\frac{-u^2}{v}\right]_{u=2}^{u=0} dv = \int_1^3 \frac{4}{v}\,dv = 4\ln v\Big]_1^3$$

$$= 4\ln 3 - 4\ln 1$$

$$= \boxed{4\ln 3}$$

11. Integrate $\displaystyle\int_C \frac{y^2\,dx + x^2\,dy}{(x+y)^2}$ where C is the curve $x = 1 + t\sin(\pi t/2)$, $y = 2t + \cos(\pi t/2)$ from $t = 0$ to $t = 1$. (Hint: Show that the integral is path-independent in an appropriate region.)

$$\frac{\partial}{\partial y}\left(\frac{y^2}{(x+y)^2}\right) = \frac{(x+y)^2\cdot 2y - y^2\cdot 2(x+y)}{(x+y)^4} = \frac{2xy + 2y^2 - 2y^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

$$\frac{\partial}{\partial x}\left(\frac{x^2}{(x+y)^2}\right) = \frac{(x+y)^2\cdot 2x - x^2\cdot 2(x+y)}{(x+y)^4} = \frac{2x^2 + 2xy - 2x^2}{(x+y)^3} = \frac{2xy}{(x+y)^3}$$

Since these partials are equal, the integral is path-Independent in any Simply connected region where these derivatives are continuous.
For example, it is path-independent in the region
$D = 1^{st}$ quadrant $= \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$.
and the curve C is contained in this region.

$$\therefore \int_C \frac{y^2\,dx + x^2\,dy}{(x+y)^2} = \int_{C'} \frac{y^2\,dx + x^2\,dy}{(x+y)^2}$$

where C' is any other curve in the first quadrant joining $C(0) = (1,1)$ and $C(1) = (2,2)$

So let C' be the straight line $y = x$:

$$= \int_1^2 \frac{x^2\,dx + x^2\,dx}{(x+x)^2} = \int_1^2 \frac{2x^2}{4x^2}\,dx = \int_1^2 \frac{1}{2}\,dx = \frac{x}{2}\Big]_1^2$$

$$= \boxed{\frac{1}{2}}.$$