Math 5B, Solutions to Final Review Problems

Fall 2006

1. Integrate $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$.

   Solution. Since the integral $\int \frac{\sin x}{x}\, dx$ is too hard, we change the order of integration so that we integrate with respect to y first. This double integral is taken over a region R, which is defined by the inequalities $0 \le y \le 1$ and $y \le x \le 1$. Graphing this region, it is clear that it is the triangle with vertices $(0,0),(1,0),(1,1)$. Thus it is also defined by the inequalities $0 \le x \le 1$ and $0 \le y \le x$. Hence

$$\begin{aligned} \int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy &= \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx \\ &= \int_0^1 \sin x\, dx \\ &= 1 - \cos 1. \end{aligned}$$

2. Integrate $\int\int_R \frac{1}{1+x^2+y^2}\, dx\, dy$ where R is the region bounded by the top half of the unit circle and the x-axis.

   Solution. Convert to Polar Coordinates by making the substitutions $x = r\cos\theta$ and $y = r\sin\theta$. R is then described by the inequalities $0 \le r \le 1$ and $0 \le \theta \le \pi$. The Jacobian $|\frac{\partial(x,y)}{\partial(r,\theta)}| = r$, and thus the integral becomes

$$\begin{aligned} \int\int_R \frac{1}{1+r^2}\, r\, dr\, d\theta &= \int_0^\pi \int_0^1 \frac{r}{1+r^2}\, dr\, d\theta \\ &= \int_0^\pi \frac{1}{2} \ln(1+r^2)]_0^1\, d\theta \\ &= \int_0^\pi \frac{1}{2} \ln 2\, d\theta \\ &= \frac{\pi}{2} \ln 2. \end{aligned}$$

3. Integrate $\int\int_R 8xy\, dx\, dy$ where R is the interior of the rectangle with vertices $(0,0)$, $(1,1)$, $(2,-2)$ and $(3,-1)$.

   Solution. Notice that the sides of the rectangle are not parallel to the x,y-axes. Thus we search for a change of variables that will simplify the integral. The sides of the rectangle have equations

- $y = x$ ('left' side from $(0,0)$ to $(1,1)$),
- $y = x - 4$ ('right' side from $(2,-2)$ to $(3,-1)$),
- $y = -x$ ('bottom' from $(0,0)$ to $(2,-2)$), and
- $y = -x + 2$ ('top' from $(1,1)$ to $(3,-1)$).

So we set $u = x - y$ and $v = x + y$. From the 'left' to 'right' u ranges from 0 to 4, and from the 'bottom' to 'top' v ranges from 0 to 2. In order to substitute something into $8xy$ we need to solve for x and y in terms of u and v. Doing so yields $x = (u+v)/2$ and $y = (v-u)/2$. So the Jacobian $|\frac{\partial(x,y)}{\partial(u,v)}| = 1/2$, and the integral becomes

$$\begin{aligned}\int_0^2 \int_0^4 8\left(\frac{u+v}{2}\right)\left(\frac{v-u}{2}\right)\left(\frac{1}{2}\right)\,du\,dv &= \int_0^2 \int_0^4 (v^2 - u^2)\,du\,dv \\ &= \int_0^2 (4v^2 - 64/3)\,dv \\ &= \frac{32}{3} - \frac{128}{3} = -32.\end{aligned}$$

4. Consider the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$ above the triangle R with vertices $(0,0), (1,0), (0,1)$ in the xy-plane.

   (a) Find the volume of the region below the surface and above the triangle R.

   Solution.

$$\begin{aligned} V &= \int\int_R \frac{2}{3}(x^{3/2} + y^{3/2})\,dx\,dy \\ &= \int_0^1 \int_0^{1-x} \frac{2}{3}(x^{3/2} + y^{3/2})\,dy\,dx \\ &= \int_0^1 \frac{2}{3}x^{3/2}(1-x) + \frac{4}{15}(1-x)^{5/2}\,dx \\ &= (\frac{4}{15}x^{5/2} - \frac{4}{21}x^{7/2} - \frac{8}{105}(1-x)^{7/2})]_0^1 \\ &= \frac{4}{15} - \frac{4}{21} + \frac{8}{105} = \frac{16}{105}.\end{aligned}$$

   (b) Find the surface area of the surface above the triangle R.

Solution.

$$\begin{aligned}
S &= \int\int_R \sqrt{1+(x^{1/2})^2+(y^{1/2})^2}\; dx\; dy \\
&= \int_0^1 \int_0^{1-x} \sqrt{1+x+y}\; dy\; dx \\
&= \int_0^1 \frac{2}{3}(1+x+y)^{3/2}]_0^{1-x}\; dx \\
&= \int_0^1 \frac{2}{3}(2^{3/2}-(1+x)^{3/2})\; dx \\
&= \frac{2}{3}(2^{3/2}-\frac{2}{5}2^{5/2}+\frac{2}{5}) = \frac{4+4\sqrt{2}}{15}.
\end{aligned}$$

5. Find the area inside the closed curve C with parametric equations $x(t) = (t-t^2)\cos(\pi t)$ and $y(t) = (t-t^2)\sin(\pi t)$ for $0 \le t \le 1$.

Solution.

$$\begin{aligned}
A &= \oint_C x\; dy \\
&= \int_0^1 (t-t^2)\cos(\pi t)[(1-2t)\sin(\pi t)+\pi(t-t^2)\cos(\pi t)]\; dt \\
&= \int_0^1 \frac{1}{2}(1-2t)(t-t^2)\sin(2\pi t)+\frac{\pi}{2}(t-t^2)^2(1+\cos(2\pi t))\; dt \\
&= \frac{1}{2}\int_0^1 [(1-2t)(t-t^2)\sin(2\pi t)+\pi(t-t^2)^2\cos(2\pi t)]\; dt + \frac{\pi}{2}\int_0^1 (t-t^2)^2\; dt,
\end{aligned}$$

where we have used the identities $\sin(\pi t)\cos(\pi t) = \frac{1}{2}\sin(2\pi t)$ and $\cos^2(\pi t) = \frac{1}{2}(1+\cos(2\pi t))$. We now recognize that the first integrand is the derivative of $\frac{1}{2}(t-t^2)^2\sin(2\pi t)$ (alternatively, if you use integration by parts on the first summand of this integral with $dv = (1-2t)(t-t^2)dt$ and $u = \sin(2\pi t)$, some nice cancellation will occur). Thus, we have

$$\begin{aligned}
A &= \frac{1}{4}((t-t^2)^2\sin(2\pi t))]_0^1 + \frac{\pi}{2}\int_0^1 (t^2-2t^3+t^4)\; dt \\
&= 0+\frac{\pi}{2}(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}) = \frac{\pi}{60}.
\end{aligned}$$

6. Evaluate the line integrals.

(a) $\int_C xy^4\ ds$ where C is the top half of the circle $x^2+y^2=4$ from $(2,0)$ to $(-2,0)$.
Solution. Use the parametrization $x = 2\cos t$ and $y = 2\sin t$ for $0 \le t \le \pi$. Then

$$\begin{aligned}\int_C xy^4\ ds &= \int_0^\pi (2\cos t)(2\sin t)^4\sqrt{(-2\sin t)^2+(2\cos t)^2}\ dt \\ &= 64\int_0^\pi \sin^4 t\cos t\ dt \\ &= \frac{64}{5}\sin^5 t]_0^\pi = 0.\end{aligned}$$

(b) $\int_C \mathbf{u}\cdot d\mathbf{r}$ where $\mathbf{u} = \frac{x}{y}\mathbf{i}+\frac{y}{x}\mathbf{j}$ and C is the (shorter) arc of the unit circle from $(\sqrt{3}/2, 1/2)$ to $(1/2, \sqrt{3}/2)$.
Solution. We parametrize C by $x=\cos t$ and $y = \sin t$ for $\pi/6 \le t \le \pi/3$. Then

$$\begin{aligned}\int_C \mathbf{u}\cdot d\mathbf{r} &= \int_C \frac{x}{y}\ dx + \frac{y}{x}\ dy \\ &= \int_{\pi/6}^{\pi/3} (\frac{\cos t}{\sin t}(-\sin t) + \frac{\sin t}{\cos t}\cos t)\ dt \\ &= \int_{\pi/6}^{\pi/3} (\sin t - \cos t)\ dt \\ &= (-\cos t - \sin t)]_{\pi/6}^{\pi/3} = -1/2 - \sqrt{3}2 + 1/2 + \sqrt{3}/2 = 0.\end{aligned}$$

7. Evaluate $\oint_C \frac{-y\ dx + x\ dy}{x^2+y^2}$ when (a) C is the unit circle traversed in the counter-clockwise direction; and (b) C is the parallelogram with vertices $(2,3), (3,5), (5,2), (6,4)$ traversed in the counter clockwise direction.

Solution. (a) Since the integrand is not continuous at $(0,0)$, which is inside the unit circle C, we cannot use Green's Theorem here and must do the integral by hand. Parametrize C by $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$.

$$\begin{aligned}\oint_C \frac{-y\ dx + x\ dy}{x^2+y^2} &= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t}\ dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

(b) Fortunately, this parallelogram does not contain the origin, and thus we can use Green's Theorem:

$$\begin{aligned}\oint_C \frac{-y\ dx + x\ dy}{x^2+y^2} &= \int\int_R \frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) + \frac{\partial}{\partial y}(\frac{y}{x^2+y^2})\ dx\ dy \\ &= \int\int_R (\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2})\ dx\ dy = 0.\end{aligned}$$

(Notice the integral in this problem will be 0 around any closed curve not containing the origin. It is thus path-independent in any region D without holes and not containing the origin.)

8. Evaluate $\oint_C y^3\ dx - x^3\ dy$ where (a) C is the unit circle traversed counter-clockwise; and (b) C is the square with vertices $(\pm 1, \pm 1)$ traversed clockwise.

Solution. (a) The integral does not appear to be path-independent ($\frac{\partial}{\partial y}(y^3) \neq \frac{\partial}{\partial x}(x^3)$), so we try Green's Theorem:

$$\oint_C y^3\ dx - x^3\ dy = \int\int_R -3x^2 - 3y^2\ dx\ dy,$$

where R is the interior of the unit circle. To simplify the integral, we convert to polar coordinates:

$$\begin{aligned}\int\int_R -3x^2 - 3y^2\ dx\ dy &= \int_0^{2\pi}\int_0^1 -3r^2 r\ dr\ d\theta \\ &= \int_0^{2\pi} -3/4\ d\theta = \frac{-3\pi}{2}.\end{aligned}$$

(b) Again using Green's Theorem, and inserting a minus sign because we are traversing the square in the clockwise direction, we have

$$\begin{aligned}\oint_C y^3\ dx - x^3\ dy &= -\int\int_R -3x^2 - 3y^2\ dx\ dy \\ &= 3\int_{-1}^1\int_{-1}^1 (x^2+y^2)\ dx\ dy \\ &= 3\int_{-1}^1 \frac{2}{3} + 2y^2\ dy \\ &= (\frac{2}{3}y - \frac{y^3}{3})]_{-1}^1 = 2.\end{aligned}$$

9. Evaluate the following integrals.

(a) $\displaystyle\int_C \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy$ where C is the parabola $y = 2 + x^2$ from $(0, 2)$ to $(1, 3)$.

Solution. Notice $\frac{3x^2}{y} = \frac{\partial}{\partial x}(\frac{x^3}{y})$ and $-\frac{x^3}{y^2} = \frac{\partial}{\partial y}(\frac{x^3}{y})$. Thus the integral is path-independent and can be evaluated by plugging in the endpoints into $F(x, y) = \frac{x^3}{y}$ and subtracting:

$$\int_C \frac{3x^2}{y}\,dx - \frac{x^3}{y^2}\,dy = F(1, 3) - F(0, 2) = \frac{1}{3}.$$

(b) $\displaystyle\int_C \sec^2 x \tan y\ dx + \sec^2 y \tan x\ dy$ where C is the curve $y = 16x^3/\pi^2$ from $(0, 0)$ to $(\pi/4, \pi/4)$.

Solution. Notice $\sec^2 x \tan y = \frac{\partial}{\partial x}(\tan x \tan y)$ and $\sec^2 y \tan x = \frac{\partial}{\partial y}(\tan x \tan y)$. Thus the integral is path-independent and we have:

$$\int_C \sec^2 x \tan y\ dx + \sec^2 y \tan x\ dy = (\tan x \tan y)]_{(0,0)}^{(\pi/4,\pi/4)} = 1.$$

(c) $\displaystyle\oint_C [\sin(xy) + xy\cos(xy)]\ dx + x^2\cos(xy)\ dy$ where C is the unit circle in the counter-clockwise direction.

Solution. We check for path-independence (or just apply Green's Theorem):

$$\frac{\partial}{\partial y}(\sin(xy) + xy\cos(xy)) = x\cos(xy) + x\cos(xy) - x^2 y\sin(xy),$$

$$\frac{\partial}{\partial x}(x^2\cos(xy)) = 2x\cos(xy) - x^2 y\sin(xy).$$

Since these are equal at every point of $\mathbb{R}^2$, the integral is path-independent, and thus it equals 0.

(d) $\displaystyle\oint_C xy^6\ dx + (3x^2y^5 + 6x)\ dy$ where C is the ellipse $x^2 + 4y^4 = 4$ traversed in the counter-clockwise direction. (Hint: the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $ab\pi$.)

Solution. We can check for path-independence again, but

$$\frac{\partial}{\partial y}(xy^6) = 6xy^5 \neq \frac{\partial}{\partial x}(3x^2y^5 + 6x) = 6xy^5 + 6,$$

so we just use Green's Theorem:

$$\begin{aligned}
\oint_C xy^6\ dx + (3x^2y^5 + 6x)\ dy &= \int\int_R (6xy^5 + 6 - 6xy^5)\ dx\ dy \\
&= 6\int\int_R dx\ dy \\
&= 6 * (\text{Area of ellipse}) = 6 * 2 * 1 * \pi = 12\pi.
\end{aligned}$$