Math 5B, Midterm 1 Review Problems - Solutions

Fall 2006

1. Consider the two planes P_1, defined by the equation $2x - y + z = 1$, and P_2, given by the equation $-x + y - z = 3$.

   (a) Find parametric equations for the line that is the intersection $P_1 \cap P_2$ of the planes P_1 and P_2.

   Solution. P_1 has normal vector $\mathbf{n_1} = (2, -1, 1)$ and P_2 has normal vector $\mathbf{n_2} = (-1, 1, -1)$ — just copy the coefficients of x, y, z in the equation of the plane to get the coordinates of the normal vector. The vector $\mathbf{v} = \mathbf{n_1} \times \mathbf{n_2}$ is orthogonal to both $\mathbf{n_1}$ and $\mathbf{n_2}$ by definition, and so it is parallel to both planes. That is, $\mathbf{v}$ is the direction vector of the line that we want.

   $$\mathbf{v} = (2, -1, 1) \times (-1, 1, -1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = \mathbf{j} + \mathbf{k} = (0, 1, 1).$$

   To get a point on the line, we can set $z = 0$ in the two equations of the planes and solve the resulting system of two equations for x and y. Putting $z = 0$ gives $2x - y = 1$ and $-x + y = 3$. Solving for x and y yields $x = 4$ and $y = 7$, so $(4, 7, 0)$ is a point on both planes, and thus on their line of intersection. Hence the equation of this line is $x(t) = 4,\ y(t) = 7 + t,\ z(t) = t$.

   (b) Does there exist a line that is perpendicular to both planes P_1 and P_2? Justify your answer.

   Solution. No. If there were such a line, its direction vector would be normal to both planes. But as shown in (a), the two planes have distinct (linearly independent) normal vectors (ie., the planes are not parallel), so this is impossible.

   (c) Find an equation of the plane that contains the point $(1, 1, 1)$ and the line L with equations $x(t) = 1 + 2t,\ y(t) = 3 - t,\ z(t) = 4t$.

   Solution. Setting $t = 0$, we get the point $(1, 3, 0)$ on the line L. Thus the direction vector $(2, -1, 4)$ of L and the vector $(1, 3, 0) - (1, 1, 1) = (0, 2, -1)$ are parallel to the desired plane, so to find a normal vector we can take their cross product:

   $$(2, -1, 4) \times (0, 2, -1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 4 \\ 0 & 2 & -1 \end{vmatrix} = -7\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} = (-7, 2, 4).$$

   Since $(1, 1, 1)$ must be in the plane, we get the equation

   $$-7(x - 1) + 2(y - 1) + 4(z - 1) = 0.$$

2. Sketch at least 5 level curves of the surface given by the equation $z = e^{x-y}$, and then sketch the surface. (Be sure to label your axes.)

   Solution. To find the graphs of the level curves, we set $z = c$ and solve for y to get $y = x - \ln c$. Thus the level curves are lines with slope 1 and y-intercept $-\ln c$ depending on the height z. Notice that these are only defined for values of $c > 0$ (ie., the surface will lie entirely above the xy-plane. Finally, to graph the whole surface, it helps to look at a couple cross sections, say by setting $x = 0$ and $x = 2$. If $x = 0$, we get the equation $z = e^{-y}$, and if $x = 2$ we get $z = e^{2-y} = e^2 e^{-y}$. See separate files for graphs.

3. Calculate the following limits, or show that they do not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1+xy}{x^2+y^2}$

   Solution. As x and y approach 0, $1 + xy$ approaches 1, while $x^2 + y^2$ approaches 0 and is always positive (you can write $x^2 + y^2 \to 0^+$). Thus the limit will be $+\infty$.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x}{y} + \frac{y}{x}\right)$

   Solution. $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x}{y} + \frac{y}{x}\right) = \lim_{(x,y)\to(0,0)} \left(\frac{x^2+y^2}{xy}\right)$. Since plugging in $(0, 0)$ yields $0/0$, we try converting to polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$. Thus

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2+y^2}{xy}\right) = \lim_{r\to 0^+} \frac{r^2}{r^2\cos\theta\sin\theta} = \frac{1}{\cos\theta\sin\theta},$$

   which depends on θ. Thus the limit does not exist.

   (Note: Even though the limit does not exist, you CANNOT prove this by saying $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x}{y} + \frac{y}{x}\right) = \lim_{(x,y)\to(0,0)} \frac{x}{y} + \lim_{(x,y)\to(0,0)} \frac{y}{x}$ and arguing that each limit in the sum does not exist, since these limits must exist in order to apply the limit law to split up the limit as a sum of two limits.)

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2}$

   Solution. Convert to polar coordinates to get

$$\lim_{r\to 0^+} \frac{e^{r^2}-1}{r^2} = \lim_{r\to 0^+} \frac{2re^{r^2}}{2r} = e^0 = 1,$$

   where we used l'Hospital's rule for the first equality.

4. Let $f(x, y) = \dfrac{2}{\frac{1}{x} + \frac{1}{y}}$.

(a) What is the domain D of f? (Write your answer in the form $\{(x, y) \in \mathbb{R}^2 \mid \dots \}$.)
Solution. Simplifying, we have $f(x, y) = \frac{2}{\frac{y+x}{xy}} = \frac{2xy}{x+y}$, which is defined as long as $x + y \neq 0$. So $D = \{(x, y) \in \mathbb{R}^2 \mid y \neq -x \}$.

(b) Find $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D}} f(x, y)$.

(Note: To compute this limit, you should only consider points (x, y) that are in the domain D of f, and not any points (x, y) near $(0, 0)$ where $f(x, y)$ is undefined.)
Solution. To find the limit (and show that it exists), we must compute the limit as (x, y) approaches $(0, 0)$ from all possible directions in D. All of these directions are accounted for by the lines with equations $y = cx$ for $c \neq -1$, and by the vertical line $x = 0$.

$$\lim_{\substack{(x,y)\to(0,0)\\y=cx}} f(x, y) = \lim_{x\to 0} \frac{2xcx}{x + cx} = \lim_{x\to 0} \frac{2cx}{1 + c} = 0,$$

since $1 + c \neq 0$. If we set $x = 0$ and take the limit as $y \to 0$, we get $\lim_{y\to 0} \frac{0}{y} = 0$. Therefore, the limit exists and is equal to 0.

5. Let $z = \frac{x-y}{x^2+y^2}$, and let $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Find dz in terms of dx and dy.
Solution. One method is to find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ separately, ant then use the definition $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$. Instead, we will differentiate the whole expression using the quotient rule (the chain rule guarantees that this works–just imagine you are differentiating with respect to t, but write dz, dx, dy for $dz/dt, dx/dt, dy/dt$). We get

$$dz = \frac{(x^2 + y^2)(dx - dy) - (x - y)(2xdx + 2ydy)}{(x^2 + y^2)^2} = \frac{y^2 - x^2 + 2xy}{(x^2 + y^2)^2}dx + \frac{y^2 - x^2 - 2xy}{(x^2 + y^2)^2}dy.$$

(b) Find dz in terms of dr and $d\theta$. (Your answer should not contain any x's or y's.)
Solution. One method is to substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the equation for z, and then proceed as in part (a). That is probably the easiest, but

we will use the chain rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ separately.

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2}\cos\theta + \frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}\sin\theta \\
&= \frac{\cos\theta(\sin^2\theta - \cos^2\theta + 2\cos\theta\sin\theta) + \sin\theta(\sin^2\theta - \cos^2\theta - 2\sin\theta\cos\theta)}{r^2}; \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{y^2 - x^2 + 2xy}{(x^2+y^2)^2}(-r\sin\theta) + \frac{y^2 - x^2 - 2xy}{(x^2+y^2)^2}(r\cos\theta) \\
&= \frac{-\sin\theta(\sin^2\theta - \cos^2\theta + 2\cos\theta\sin\theta) + \cos\theta(\sin^2\theta - \cos^2\theta - 2\sin\theta\cos\theta)}{r}
\end{aligned}$$

Finally, substitute these into $dz = \frac{\partial z}{\partial r}dr + \frac{\partial z}{\partial \theta}d\theta$.

(c) Find $(\frac{\partial z}{\partial r})_\theta$ and $(\frac{\partial z}{\partial r})_x$ (for the second, assume $y > 0$).

Solution. $(\frac{\partial z}{\partial r})_\theta$ is just $\frac{\partial z}{\partial r}$ calculated above in (b). To find $(\frac{\partial z}{\partial r})_x$, we must express z as a function of only r and x:

$$z = \frac{x-y}{x^2+y^2} = \frac{x-y}{r^2} = \frac{x - \sqrt{r^2 - x^2}}{r^2}.$$

Now we differentiate with respect to r:

$$(\frac{\partial z}{\partial r})_x = \frac{-r^2(r^2-x^2)^{-1/2}r - (x - (r^2-x^2)^{1/2})2r}{r^4}.$$

(d) Approximate the value of z (using (a)) when $x = 1.01$ and $y = 0.98$.

Solution. Let $\Delta z = z(1.01, 0.98) - z(1,1)$. We know that dz at $(1,1)$ approximates Δz, so we have $\Delta z \approx dz|_{(1,1)} = \frac{2}{4}dx + \frac{-2}{4}dy = (.01)/2 - (-.02)/2 = .015$, where we have used $dx = 1.01 - 1 = .01$ and $dy = .98 - 1 = -.02$. Hence

$$z(1.01, 0.98) = z(1,1) + \Delta z \approx z(1,1) + .015 = 0 + .015 = .015.$$

6. Suppose we have functions $\mathbf{z}(y_1, y_2, y_3) : \mathbb{R}^3 \to \mathbb{R}^3$ and $\mathbf{y}(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\mathbf{z} = \left\{ \begin{array}{lcl} z_1 &=& y_1 y_2 - 2y_3 e^{y_2} \\ z_2 &=& y_2\cos y_1 - y_1 \sin y_2 + y_1 y_2 y_3^2 \\ z_3 &=& y_1^3 + y_2^3 + y_3^3 \end{array} \right. \quad \text{and} \quad \mathbf{y} = \left\{ \begin{array}{lcl} y_1 &=& x_1^2 - x_2^2 \\ y_2 &=& 2x_1x_2 \\ y_3 &=& x_1 + x_2 \end{array} \right.$$

Write the Jacobian matrix of the composition $\mathbf{z} \circ \mathbf{y}$ as a product of two matrices (do not evaluate this product), and compute $\frac{\partial z_2}{\partial x_1}$.

Solution. The chain rule says that the Jacobian $\mathbf{z_x} = \mathbf{z_y}\mathbf{y_x}$. Thus,

$$\mathbf{z_x} = \begin{pmatrix} y_2 & y_1 - 2y_3e^{y_2} & -2e^{y_2} \\ -y_2\sin y_1 - \sin y_2 + y_2y_3^2 & \cos y_1 - y_1\cos y_2 + y_1y_3^2 & 2y_1y_2y_3 \\ 3y_1^2 & 3y_2^2 & 3y_3^2 \end{pmatrix} \begin{pmatrix} 2x_1 & -2x_2 \\ 2x_2 & 2x_1 \\ 1 & 1 \end{pmatrix}.$$

To get $\frac{\partial z_2}{\partial x_1}$ we multiply the second row of $\mathbf{z_y}$ by the first column of $\mathbf{y_x}$:

$$\frac{\partial z_2}{\partial x_1} = (-y_2 \sin y_1 - \sin y_2 + y_2 y_3^2)2x_1 + (\cos y_1 - y_1 \cos y_2 + y_1 y_3^2)2x_2 + 2y_1 y_2 y_3.$$

Ideally, we should now use the definition of $\mathbf{y}$ to convert the y_i's into functions of x_1 and x_2, but that will take too much space to write down.

7. Let $f(u)$ be a function of a single variable u, and define $z(x, y) = f(ax + by)$ where a and b are fixed real numbers. Show that

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0.$$

Solution. (It should be stated in the problem that $f(u)$ is a differentiable function.) Notice, $z = f(u)$ where $u(x, y) = ax + by$. By the chain rule, we have $\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = a\frac{dz}{du}$, and $\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = b\frac{dz}{du}$. Thus,

$$b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = b(a\frac{dz}{du}) - a(b\frac{dz}{du}) = 0.$$

8. Suppose a function $f(x, y)$ of two variables satisfies the law $f(tx, ty) = tf(x, y)$ for all values of x, y and t in $\mathbb{R}$. Show that $xf_x + yf_y = f(x, y)$ for all values of x and y.

Solution. (It should be stated in the problem that $f(x, y)$ is differentiable on all of $\mathbb{R}^2$.)Imagine that x and y are fixed and differentiate the equation $f(tx, ty) = tf(x, y)$ with respect to t. On the left, using the chain rule, we have $\frac{df(tx,ty)}{dt} = f_x(tx, ty)(\frac{\partial(tx)}{\partial t}) + f_y(tx, ty)(\frac{\partial(ty)}{\partial t}) = f_x(tx, ty)x + f_y(tx, ty)y$, while differentiating the right hand side with respect to t yields simply $\frac{d}{dt}(tf(x, y)) = f(x, y)$. Hence, we have an equality of functions of t: $xf_x(tx, ty) + yf_y(tx, ty) = f(x, y)$. Setting $t = 1$, gives $xf_x(x, y) + yf_y(x, y) = f(x, y)$, and this holds for any values of x and y.