Math 5B, Midterm 2 Solutions

Fall 2006

1. Suppose $u = \cos x + y$ and $v = \sin y - x$. Find $\left(\frac{\partial x}{\partial u}\right)_v$ and $\left(\frac{\partial y}{\partial v}\right)_u$.

Solution. The Jacobian matrix $\frac{\partial(x,y)}{\partial(u,v)}$ is simply the inverse of the Jacobian matrix $\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} -\sin x & 1 \\ -1 & \cos y \end{pmatrix}$. The determinant of this matrix is $-\sin x \cos y + 1$, and thus its inverse is $\begin{pmatrix} \frac{\cos y}{1-\sin x \cos y} & \frac{-1}{1-\sin x \cos y} \\ \frac{1}{1-\sin x \cos y} & \frac{-\sin x}{1-\sin x \cos y} \end{pmatrix}$. Thus

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{\cos y}{1 - \sin x \cos y}$$

is the upper left entry of this Jacobian, while

$$\left(\frac{\partial y}{\partial v}\right)_u = \frac{-\sin x}{1 - \sin x \cos y}$$

is the lower right entry.

2. Let S be the surface given by the equation $x^2 + yz = 1$, and let C be the curve defined by the equations $\begin{cases} x(t) &= \cos t - \sin t \\ y(t) &= 2 \sin t \\ z(t) &= \cos t \end{cases}$ for $0 \leq t \leq 2\pi$.

   (a) Show that the curve C is contained in the surface S.

   Solution. Let $F(x, y, z) = x^2 + yz - 1$, so that we must check that $F(\cos t - \sin t, 2 \sin t, \cos t) = 0$. We have $F(\cos t - \sin t, 2 \sin t, \cos t) = (\cos t - \sin t)^2 + 2 \sin t \cos t - 1 = \cos^2 t - 2 \sin t \cos t + \cos^2 t + 2 \sin t \cos t - 1 = \cos^2 t + \sin^2 t - 1 = 0$.

   (b) Find parametric equations for the tangent line to C at the point where $t = \pi/4$.

   Solution. The tangent vector to C at $t = \pi/4$ is given by $(x'(\pi/4), y'(\pi/4), z'(\pi/4)) = (-\sin(\pi/4) - \cos(\pi/4), 2\cos(\pi/4), -\sin(\pi/4)) = (-\sqrt{2}, \sqrt{2}, -\sqrt{2}/2)$ (we could also use the tangent vector $(-2, 2, -1)$ obtained by multiplying this one by the scalar $\sqrt{2}$). The coordinates of the point on C when $t = \pi/4$ are $(x(\pi/4), y(\pi/4), z(\pi/4)) = (0, \sqrt{2}, \sqrt{2}/2)$. Thus the parametric equation of the tangent line is

$$\begin{cases} x(t) &= -t\sqrt{2} \\ y(t) &= \sqrt{2} + t\sqrt{2} \\ z(t) &= \sqrt{2}/2 - t\sqrt{2}/2. \end{cases}$$

(c) Find the equation of the tangent plane to S at the point $(3, -2, 4)$.

Solution. The normal vector is given by the gradient of F. Thus $\mathbf{n} = \nabla F|_{(3,-2,4)} = (2x, z, y)|_{(3,-2,4)} = (6, 4, -2)$. So the equation of the tangent plane is $6(x-3) + 4(y+2) - 2(z-4) = 0$.

3. A bee is flying through a region of space where the temperature at a point (x, y, z) is given by the function $T(x, y, z) = \frac{z}{x+y^2} + zy$

(a) If the bee is at $(2, 1, 3)$ and is too cold, which direction should the bee fly to get warmer as quickly as possible? (Give your answer as a vector.)

Solution. The gradient vector $\nabla T|_{(2,1,3)}$ points in the direction in which the temperature is increasing most rapidly. We have $\nabla T|_{(2,1,3)} = (\frac{-z}{(x+y^2)^2}, \frac{-2yz}{(x+y^2)^2} + z, \frac{1}{x+y^2} + y)|_{(2,1,3)} = (-1/3, 7/3, 4/3)$. (Any positive multiple of this vector, for instance $(-1, 7, 4)$, would also be correct.)

(b) If the bee is flying along a curve $\mathbf{r}(t) = (2t^3, t, 1-t^2)$, is the bee getting colder or warmer when $t = 1$? Justify your answer.

Solution. When $t = 1$, the bee is at $\mathbf{r}(1) = (2, 1, 0)$ with velocity vector $\mathbf{r}'(t) = (6t, 1, -2t)|_{t=1} = (6, 1, -2)$. The instantaneous rate of change of the temperature T along this curve at $t = 1$ is given by the directional derivative $\nabla_{(6,1,-2)} T|_{(2,1,0)} = \frac{(6,1,-2)}{|(6,1,-2)|} \cdot \nabla T|_{(2,1,0)}$. From (a), $\nabla T|_{(2,1,0)} = (\frac{-z}{(x+y^2)^2}, \frac{-2yz}{(x+y^2)^2} + z, \frac{1}{x+y^2} + y)|_{(2,1,0)} = (0, 0, 4/3)$. Thus $\nabla_{(6,1,-2)} T|_{(2,1,0)} = \frac{1}{\sqrt{41}}(6, 1, -2) \cdot (0, 0, 4/3) = \frac{-8}{3\sqrt{41}} < 0$, so the temperature is decreasing and the bee is getting colder.

4. Let $f(x, y) = x^3 - x^2 + 2xy + y^2$. Find all critical points of $f(x, y)$ and classify each as a relative maximum, relative minimum or saddle point.

Solution. We set the gradient of f equal to 0: $\nabla f = (3x^2 - 2x + 2y, 2x + 2y) = (0, 0)$. The second equation, $2x + 2y = 0$, implies that $y = -x$. Substituting this into the first equation gives $3x^2 - 4x = x(3x-4) = 0$. Hence $x = 0$ (and $y = -x = 0$), or else $x = 4/3$ (and $y = -x = -4/3$). Thus there are two critical points $(0, 0)$ and $(4/3, -4/3)$. The Hessian matrix of second order partial derivatives of $f(x, y)$ is $\begin{pmatrix} 6x-2 & 2 \\ 2 & 2 \end{pmatrix}$. At $(0, 0)$ we get the matrix $\begin{pmatrix} -2 & 2 \\ 2 & 2 \end{pmatrix}$, which has determinant $-2 * 2 - 2 * 2 = -8 < 0$. Thus $(0, 0)$ is a saddle point. At $(4/3, -4/3)$, the matrix becomes $\begin{pmatrix} 6 & 2 \\ 2 & 2 \end{pmatrix}$, which has determinant $6*2-2*2 = 8 > 0$ and trace $6+2 = 8 > 0$, so $(4/3, -4/3)$ is a relative minimum.

5. Find the maximum value of $f(x, y) = (x + 3)y$ on the set $\{(x, y) \mid x^2 + 4y^2 \leq 9\ \}$.

   Solution. We first look for critical points: $\nabla f = (y, x+3) = (0, 0)$ implies that $x = -3$ and $y = 0$, so $(-3, 0)$ is the only critical point. Now we use Lagrange multipliers to find where the extrema occur on the boundary $x^2+4y^2 = 9$. Let $g(x, y) = x^2+4y^2-9$. We must solve the $\nabla f = (y, x + 3) = \lambda\nabla g = \lambda(2x, 8y)$. We thus have a total of 3 equations relating x, y, λ:

   (1) $y = 2\lambda x$.

   (2) $x + 3 = 8\lambda y$.

   (3) $x^2 + 4y^2 = 9$.

   First notice that x cannot be 0, since then (1) would imply that $y = 0$, but $(0, 0)$ does not satisfy (3). Similarly y cannot be 0. Thus, we can solve (1) and (2) for λ and set the results equal to each other:

$$\frac{y}{2x} = \lambda = \frac{x + 3}{8y}.$$

   Cross multiplying gives the relation $8y^2 = 2x(x + 3)$, or $4y^2 = x(x + 3)$. We can now substitute this into (3) to get $x^2 + x(x + 3) = 9$ and solve for x. Simplifying, we have $2x^2 + 3x - 9 = (2x - 3)(x + 3) = 0$. Hence $x = -3$ or $x = 3/2$. If $x = -3$, $y = \sqrt{(9 - x^2)/4} = 0$, giving the critical point $(-3, 0)$ again. If $x = 3/2$, $y = \pm\sqrt{(9 - 9/4)/4} = \pm\sqrt{27/16} = \pm\frac{3\sqrt{3}}{4}$. We now evaluate $f(x, y) = (x + 3)y$ at each of these 3 points to find its maximum value.

   $f(-3, 0) = 0, \qquad f(3/2, 3\sqrt{3}/4) = 9/2(3\sqrt{3}/4) = 27\sqrt{3}/8, \qquad f(3/2, -3\sqrt{3}/4) = -27\sqrt{3}/8.$

   Thus the maximum value of $f(x, y)$ is $27\sqrt{3}/8$.

6. Let $\mathbf{v} = (x + y^2)\mathbf{i} + (y - xz)\mathbf{j} + (z^3 + 2xy)\mathbf{k}$ be a vector field on $\mathbb{R}^3$.

   (a) Compute div($\mathbf{v}$).

   Solution. $\text{div}(\mathbf{v}) = \frac{\partial}{\partial x}(x+y^2)+\frac{\partial}{\partial y}(y-xz)+\frac{\partial}{\partial z}(z^3+2xy) = 1+1+3z^2 = 2+3z^2$.

   (b) Compute curl($\mathbf{v}$).

Solution.

$$\begin{aligned}
\text{curl}(\mathbf{v}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y-xz & z^3+2xy \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(z^3+2xy) - \frac{\partial}{\partial z}(y-xz)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(x+y^2) - \frac{\partial}{\partial x}(z^3+2xy)\right)\mathbf{j} + \\
&\quad \left(\frac{\partial}{\partial x}(y-xz) - \frac{\partial}{\partial y}(x+y^2)\right)\mathbf{k} \\
&= 3x\mathbf{i} - 2y\mathbf{j} - (2y+z)\mathbf{k}.
\end{aligned}$$

(c) Is $\mathbf{v} = \nabla f$ for some differentiable function $f(x, y, z)$? Justify your answer.

Solution. No. If $\mathbf{v} = \nabla f$, then $\text{curl}(\mathbf{v}) = \text{curl}(\nabla f) = 0$. But we showed in part (b) that $\text{curl}(\mathbf{v}) \neq 0$.