Math 5B - Midterm 1 Solutions

1. (a) Find parametric equations for the line that passes through the point $(2, 0, -1)$ and is perpendicular to the plane with equation $4x - y - 2z = 1$.

   Solution. The direction vector for this line is $\mathbf{v} = (4, -1, -2)$ and it must pass through the point $(2, 0, -1)$. Thus we have parametric equations $(x, y, z) = (2, 0, -1) + (4, -1, -2)t = (2 + 4t, -t, -1 - 2t)$.

   (b) Find the equation of the unique plane that contains the two lines, L_1 and L_2, whose equations are:

$$L_1 : \begin{cases} x &= t \\ y &= 2 - t \\ z &= 3 \end{cases}, \quad L_2 : \begin{cases} x &= -1 + 2t \\ y &= 3 - 2t \\ z &= 3t. \end{cases}$$

   Solution. Since the plane contains the two lines, their direction vectors $(1, -1, 0)$ and $(2, -2, 3)$ are parallel to the plane. Hence their cross product will be a normal vector.

$$\mathbf{n} = (1, -1, 0) \times (2, -2, 3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 2 & -2 & 3 \end{vmatrix} = -3\mathbf{i} - 3\mathbf{j} + 0\mathbf{k} = (-3, -3, 0).$$

   To get a point in the plane, we can take any point in either line, so just set $t = 0$ in the equations for L_1 to get the point $(0, 2, 3)$. The equation for the plane is then $-3(x - 0) - 3(y - 2) + 0(z - 3) = 0$, or more simply, $-3x - 3y + 6 = 0$.

2. Graph at least 5 level curves of $z = y^2/x$ (label them with the corresponding values of z), and then graph the surface $z = y^2/x$ for $z \geq 0$. Be sure to label your axes.

   Solution. Setting $z = c \neq 0$, we can solve for x to get $x = y^2/c$. These graphs are (sideways) parabolas in the xy-plane, that get less steep as c gets large, and more steep as c approaches 0. If $z = 0$, the level curve is $y = 0$, or just the x-axis. It is important that these level curves all have a hole where $x = 0$ since z is not defined there. (See the link to the level curves picture.) In graphing the whole surface, we sketch two cross sections for $x = 1$ and $x = 2$. These again are parabolas with equations $z = y^2$ and $z = y^2/2$, so they get less steep as x gets larger. Note that the z-axis is not actually part of the graph, again since z is not defined when $x = 0$. (See the link to the surface picture.)

3. Calculate the following limits, or show that they do not exist.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^{2xy-y^2}}{x^2 + y^2}$

   Solution. As $(x, y) \to (0, 0)$, the numerator approaches $e^0 = 1$, since it is a continuous function of x and y, while the denominator approaches 0 and is always positive. Thus the ratio approaches $+\infty$.

(b) $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^2}$

Solution. Convert to polar coordinates to get

$$\lim_{r\to 0^+} \frac{2r^3\cos\theta\sin^2\theta}{r^2} = \lim_{r\to 0^+} 2r\cos\theta\sin^2\theta = (2\cos\theta\sin^2\theta)\lim_{r\to 0^+} r = 0.$$

(c) $\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{2x-y}$

Solution. We take the limit as (x,y) approaches $(0,0)$ along two different lines with equations $y = cx$. First, let (x,y) approach $(0,0)$ along the line $y = 0$. We get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{\sin(x+y)}{2x-y} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

using l'Hospital's rule for the second equality. If (x,y) approaches $(0,0)$ along the line $y = x$ instead, we get

$$\lim_{\substack{(x,y)\to(0,0)\\ y=s}} \frac{\sin(x+y)}{2x-y} = \lim_{x\to 0} \frac{\sin(2x)}{x} = \lim_{x\to 0} \frac{2\cos(2x)}{1} = 2,$$

using l'Hospital's rule for the second equality. Since we get different limits depending on the direction from which we approach $(0,0)$, the limit does not exist.

4. A function $\mathbf{y} = (y_1, y_2)$ is defined by $y_1 = 3x_1^2 + x_2^2$ and $y_2 = \dfrac{x_1x_2 - 1}{x_1 + 2}$.

(a) Find the Jacobian matrix $\left(\dfrac{\partial y_i}{\partial x_j}\right)$, and say where $\mathbf{y}$ is differentiable.

Solution.

$$\mathbf{y}_{\mathbf{x}} = \left(\frac{\partial y_i}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 6x_1 & 2x_2 \\ \frac{2x_2+1}{(x_1+2)^2} & \frac{x_1}{x_1+2} \end{pmatrix}.$$

Each partial derivative appearing here is a rational function and thus continuous on its domain. The only time any of them are undefined is when $x_1 = -2$. Thus $\mathbf{y}$ is differentiable on the set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \neq -2\}$.

(b) Approximate $\mathbf{y}(-1.01, 2.02)$.

Solution. Let $\Delta\mathbf{y} = \mathbf{y}(-1.01, 2.02) - \mathbf{y}(-1, 2)$. We know that $d\mathbf{y}$ at $(-1, 2)$ approximates $\Delta\mathbf{y}$, so we have

$$\Delta\mathbf{y} \approx d\mathbf{y}|_{(-1,2)} = \mathbf{y}_{\mathbf{x}}|_{(-1,2)} d\mathbf{x} = \begin{pmatrix} -6 & 4 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} -0.01 \\ 0.02 \end{pmatrix} = \begin{pmatrix} 0.14 \\ -0.07 \end{pmatrix}.$$

Here we have taken $dx_1 = -1.01 - -1 = -0.01$ and $dx_2 = 2.02 - 2 = 0.02$. Thus

$$\mathbf{y}(-1.01, 2.02) = \mathbf{y}(-1, 2) + \Delta\mathbf{y} \approx (7, -3) + (0.14, -0.07) = (7.14, -3.07).$$

5. Suppose we have functions $\mathbf{z}(y_1, y_2, y_3) : \mathbb{R}^3 \to \mathbb{R}^2$ and $\mathbf{y}(x_1, x_2) : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^3$ given by

$$\mathbf{z} = \begin{cases} z_1 &= y_1y_2y_3 \\ z_2 &= y_1y_2^2 + 2y_2y_3^2 \end{cases} \quad \text{and} \quad \mathbf{y} = \begin{cases} y_1 &= x_2\cos(\pi x_1) \\ y_2 &= x_2\sin(\pi x_1) \\ y_3 &= \ln(x_1^2 + x_2^2) \end{cases}$$

(a) Express the Jacobian matrix $\left(\dfrac{\partial z_i}{\partial x_j}\right)$ of the composition $\mathbf{z} \circ \mathbf{y}$ as a product of two matrices (do not evaluate this product).

Solution. The chain rule says $\mathbf{z_x} = \mathbf{z_y}\mathbf{y_x}$. So

$$\mathbf{z_x} = \begin{pmatrix} y_2y_3 & y_1y_3 & y_1y_2 \\ y_2^2 & 2y_1y_2 + 2y_3^2 & 4y_2y_3 \end{pmatrix} \begin{pmatrix} -\pi x_2\sin(\pi x_1) & \cos(\pi x_1) \\ \pi x_2\cos(\pi x_1) & \sin(\pi x_1) \\ \frac{2x_1}{x_1^2+x_2^2} & \frac{2x_2}{x_1^2+x_2^2} \end{pmatrix}.$$

(b) Find $\left(\dfrac{\partial z_2}{\partial x_1}\right)_{x_2}(3, -1)$ and simplify your answer.

Solution. To get $\left(\dfrac{\partial z_2}{\partial x_1}\right)_{x_2}$ we must multiply the second row of the first matrix above by the first column of the second matrix, and then we need to evaluate at $(x_1, x_2) = (3, -1)$. We get

$$\left(\frac{\partial z_2}{\partial x_1}\right)_{x_2} = y_2^2(-\pi x_2\sin(\pi x_1)) + (2y_1y_2 + y_3^2)(\pi x_2\cos(\pi x_1)) + 4y_2y_3(\frac{2x_1}{x_1^2 + x_2^2}).$$

Notice that we have $y_1(3, -1) = -\cos(3\pi) = 1, y_2(3, -1) = -\sin(3\pi) = 0$ and $y_3(3, -1) = \ln 10$. Thus plugging in $x_1 = 3$ and $x_2 = -1$ in the above, the first and last terms of the sum become 0, and we are left with $y_3^2\pi x_2\cos(\pi x_1) = (\ln 10)^2\pi(-1)\cos(3\pi) = \pi(\ln 10)^2$.

6. Suppose $f(x, y)$ and $g(x, y)$ are differentiable functions, and define $h(x, y) = f(x, y)g(x, y)$. Show that $dh = fdg + gdf$.

Solution. By definition, $dh = h_x dx + h_y dy$. By the product rule, $h_x = \frac{\partial}{\partial x}(fg) = f_x g + fg_x$ and $h_y = \frac{\partial}{\partial y}(fg) = f_y g + fg_y$. Thus, substituting these expressions into the equation for dh and rearranging the terms, we have

$$dh = (f_x g + fg_x)dx + (f_y g + fg_y)dy = g(f_x dx + f_y dy) + f(g_x dx + g_y dy) = gdf + fdg.$$