Math 5B Summer 2006 Final

Name: ______________________________

Perm: ______________________________

Your work must be neat and complete in order to receive full credit

You must label all axes on all graphs for full credit.

All answers should be left in exact form; only exact decimals are fine. For instance, 1/10 and 0.1 are both acceptable, but 3.14 is not if the answer is π

If you need more space for work, use the back of the paper, but indicate on the front that there is work on the back

(1) (15 points) State which of the following do not exist, then compute any which do exist

(a) grad $(x\sin(xy) + z^2)$

(b) $\nabla \times (x^2y, z^2x, y^3)$

(c) $\nabla(t^2+1, sin(t), e^t)$

(d) div $(x\sin(xy) + z^2)$

(e) curl $(x\sin(xy) + z^2)$

(f) $\nabla \ \ (x^2y\vec{i} + z^2x\vec{j} + y^3\vec{k})$

$\boxed{\text{c, d, and e do not exist.}}$

a) $\boxed{(\sin(xy) + xy\cos(xy))\vec{i} + x^2\cos(xy)\vec{j} + 2z\vec{k}}$

b)
$$\left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2y & z^2x & y^3 & x^2y & z^2x \end{array}\right.$$

$$= \frac{\partial}{\partial y} y^3 \vec{i} + \frac{\partial}{\partial z} x^2y\vec{j} + \frac{\partial}{\partial x} z^2x\vec{k}$$
$$- \left(\frac{\partial}{\partial x} y^3\vec{j} + \frac{\partial}{\partial z} z^2x\vec{i} + \frac{\partial}{\partial y} x^2y\vec{k}\right)$$

$$= 3y^2\vec{i} + z^2\vec{k} - 2zx\vec{i} - x^2\vec{k}$$

$$= \boxed{(3y^2 - 2zx)\vec{i} + (z^2 - x^2)\vec{k}}$$

f) $2xy + 0 + 0 = \boxed{2xy}$

(2) (20 points) Find any absolute maxima and absolute minima of $f(x, y) = x^4 + y^4$ for $x^2 + y^2 \le 4$

Critical points: $\frac{\partial f}{\partial x} = 4x^3 \qquad \frac{\partial f}{\partial y} = 4y^3$

so only critical point is $(0,0)$.

boundary critical points: $4x^3 = 2\lambda x \qquad 4y^3 = 2\lambda y$

$x^2+y^2=4$

$4x^3 - 2\lambda x = 0 \Rightarrow 2x(2x^2-\lambda)=0 \Rightarrow x=0$ or $\cancel{x = \pm\sqrt{\frac{\lambda}{2}}}$ $x^2 = \frac{\lambda}{2}$

$4y^3 - 2\lambda y = 0 \Rightarrow 2y(2y^2-\lambda)=0 \Rightarrow y=0$ or $y^2 = \frac{\lambda}{2}$

$x=0,\ y=0$ does not satisfy $x^2+y^2=4$ so that combination doesn't work

If $x=0$ $y=\pm 2$ since $x^2+y^2=4$

If $y=0$ $x=\pm 2$ since $x^2+y^2=4$

If $x^2=\frac{\lambda}{2}$ and $y^2=\frac{\lambda}{2}$, then $\frac{\lambda}{2}+\frac{\lambda}{2}=4 \Rightarrow \lambda=4$

and $x=\pm\sqrt{2}$ $y=\pm\sqrt{2}$. This gives 8 boundary critical points: $(0,2), (0,-2), (2,0), (-2,0), (\sqrt{2},\sqrt{2}), (\sqrt{2},-\sqrt{2}),$ $(-\sqrt{2},\sqrt{2})$ and $(-\sqrt{2},-\sqrt{2})$.

At $(0,\pm 2)$ and $(\pm 2, 0)$ $\quad f(x,y)=16$

at $(\pm\sqrt{2}, \pm\sqrt{2})$ $\quad f(x,y)=8$

at $(0,0)$ $\quad f(x,y)=0$

So the absolute max is 16 at $(0,\pm 2)$ and $(\pm 2, 0)$ and the absolute min is 0 at $(0,0)$

(3) (15 points) Find the tangent plane and normal line to $x^2 + y^2 + z^2 = 9$ at (2,2,1)

At (x,y,z) the normal vector is $(2x, 2y, 2z)$

so at $(2,2,1)$ it is $(4,4,2)$

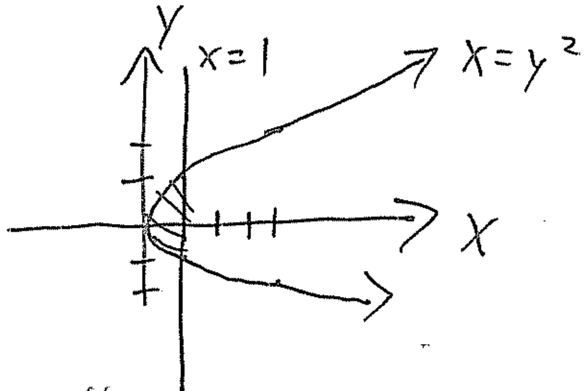
so the tangent plane is

$$\boxed{4(x-2) + 4(y-2) + 2(z-1) = 0}$$

and the normal line is:

$$\boxed{\begin{aligned} x &= 2 + 4t \\ y &= 2 + 4t \\ z &= 1 + 2t \end{aligned}}$$

(4) (20 points) (a) (5 points) Sketch the region $R = \{(x, y) \in \mathbb{R}^2 | x \geq y^2,\ x \leq 1\}$ Remember to label your axes



(b) (5 points) Convert $\iint_R 2x\, dA$ to an iterated integral of type I

$$\boxed{\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} 2x\, dy\, dx}$$

(c) (5 points) Convert $\iint_R 2x\, dA$ to an iterated integral of type II

$$\boxed{\int_{-1}^1 \int_{y^2}^1 2x\, dx\, dy}$$

(d) (5 points) What is the volume beneath the plane z=2x for $x \geq y^2$, $x \leq 1$?

Just compute either (b) or (c)

$$\int_{-1}^1 \int_{y^2}^1 2x\, dx\, dy = \int_{-1}^1 x^2 \Big|_{y^2}^1 dy = \int_{-1}^1 1^2 - (y^2)^2\, dy$$

$$= \int_{-1}^1 1 - y^4\, dy = y - \frac{y^5}{5}\Big|_{-1}^1 = 1 - \frac{1}{5} - \left(-1 - \frac{-1}{5}\right)$$

$$= \boxed{\frac{8}{5}}$$

(5) (15 points) Let $x(u,v) = 2uv$, $w(u,v) = u^2v^2$, $y(u,v) = u^2 - v^2$, and $z(u,v) = u + v$

(a) (10 points) Find $\left.\frac{\partial(w,y)}{\partial(x,z)}\right|_{(4,3)}$ I mean a matrix here, NOT its determinant.

If $u=2$, $v=1$, then $x=4$ and $z=3$, so

$$\left.\frac{\partial(w,y)}{\partial(x,z)}\right|_{(4,3)} = \left.\frac{\partial(w,y)}{\partial(u,v)}\right|_{(2,1)} \left.\frac{\partial(u,v)}{\partial(x,z)}\right|_{(4,3)}$$

$$\frac{\partial(w,y)}{\partial(u,v)} = \begin{bmatrix} 2uv^2 & 2vu^2 \\ 2u & -2v \end{bmatrix} \qquad \left.\frac{\partial(w,y)}{\partial(u,v)}\right|_{(2,1)} = \begin{bmatrix} 4 & 8 \\ 4 & -2 \end{bmatrix}$$

$$\left[\left.\frac{\partial(x,z)}{\partial(u,v)}\right|_{(2,1)}\right]^{-1} = \left.\frac{\partial(u,v)}{\partial(x,z)}\right|_{(4,3)} \qquad \frac{\partial(x,z)}{\partial(u,v)} = \begin{bmatrix} 2v & 2u \\ 1 & 1 \end{bmatrix}$$

$$\left.\frac{\partial(x,z)}{\partial(u,v)}\right|_{(2,1)} = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \qquad \left.\frac{\partial(u,v)}{\partial(x,z)}\right|_{(4,3)} = \frac{1}{2-4}\begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix} \text{ so } \left.\frac{\partial(w,y)}{\partial(x,z)}\right|_{(4,3)} = \begin{bmatrix} 4 & 8 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 0 \\ -3 & 12 \end{bmatrix}}$$

(c) (5 points) Approximate u when $x = 3.8$ and $z = 3.1$

From above $\cancel{\frac{\partial(x,z)}{\partial(u,v)}}$ $\left.\frac{\partial(u,v)}{\partial(x,z)}\right|_{(4,3)} = \begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$,

$$\text{so } \begin{bmatrix} u \\ v \end{bmatrix} \approx \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} x-4 \\ z-3 \end{bmatrix}$$

$$u(3.8, 3.1) \approx 2 + -\frac{1}{2}(3.8-4) + 2(3.1-3)$$

$$= 2 - \frac{1}{10} + \frac{2}{10} = \boxed{2.1}$$

(6) (15 points) (a) (5 points) Compute $\oint_C y\,dy$ where C is the unit circle traced once in the counterclockwise direction

$x = \cos(t)$

$y = \sin(t)$ $\quad 0 \le t \le 2\pi$

$dy = \cos(t)\,dt$

$\int_0^{2\pi} \sin(t)\cos(t)\,dt$ $\quad u = \sin(t)$, $du = \cos(t)\,dt$

$\int_0^0 u\,du = \boxed{0}$

(b) (10 points) Compute $\int_{C(0,0)}^{(1,1)} x\,ds$ where C is the line $y = x$.

$x = t$

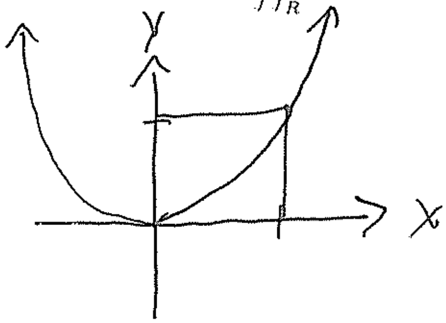
$y = t$

$0 \le t \le 1$

$ds = \sqrt{1^2 + 1^2}\,dt$

$\int_0^1 t\sqrt{2}\,dt = \sqrt{2}\,\frac{t^2}{2}\Big|_0^1 = \boxed{\frac{\sqrt{2}}{2}}$

Extra Credit: Compute $\iint_R |y - x^2| \, dA$ where $R = \{(x, y) \in \mathbb{R}^2 : 0 \le x \le 1, \ 0 \le y \le 1\}$



for $y \ge x^2$ $\quad |y-x^2| = y - x^2$

for $y < x^2$ $\quad |y-x^2| = x^2 - y$

so $\displaystyle\iint_R |y-x^2| \, dA = \int_0^1 \int_0^1 |y-x^2| \, dy \, dx$

$$= \int_0^1 \int_0^{x^2} x^2 - y \, dy \, dx + \int_0^1 \int_{x^2}^1 y - x^2 \, dy \, dx$$

$$= \int_0^1 x^2 y - \frac{y^2}{2} \Big|_0^{x^2} dx + \int_0^1 \frac{y^2}{2} - x^2 y \Big|_{x^2}^1 dx$$

$$= \int_0^1 x^2(x^2) - \frac{(x^2)^2}{2} - 0 \, dx + \int_0^1 \frac{1^2}{2} - x^2(1) - \left(\frac{(x^2)^2}{2} - x^2(x^2)\right) dx$$

$$= \int_0^1 \frac{x^4}{2} dx + \int_0^1 \frac{1}{2} - x^2 + \frac{x^4}{2} \, dx$$

$$= \frac{x^5}{10}\Big|_0^1 + \frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10}\Big|_0^1 = \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10}$$

$$= \frac{3}{30} + \frac{15}{30} - \frac{10}{30} + \frac{3}{30}$$

$$= \boxed{\frac{11}{30}}$$