1. [22 pts] (a) Find the general solution for the differential equation $y'' + 9y = 0$.

$r^2 + 9 = 0$

$r = \pm 3i$

$$\boxed{y = c_1 \cos 3t + c_2 \sin 3t}$$

(b) Solve the initial value problem $y'' + 9y = 0, \quad y(0) = 2, \quad y'(0) = -3$.

$2 = y(0) = c_1$

$y' = -3c_1 \sin 3t + 3c_2 \cos 3t$

$-3 = y'(0) = 3c_2 \implies c_2 = -1$

$$\boxed{y = 2\cos 3t - \sin 3t}$$

(c) What is the amplitude of the solution to part (b)?

$A = \sqrt{4+1} = \boxed{\sqrt{5}}$

2. [27 pts] Give answers to these problems in **real form** (that is, eliminating imaginary expressions).

(a) Find the general solution for $9y'' + 6y' + y = 0$.

$$9r^2 + 6r + 1 = 0$$
$$(3r+1)^2 = 0$$
$$r = -\frac{1}{3}, \text{ repeated}$$
$$\boxed{y = c_1 e^{-t/3} + c_2 t e^{-t/3}}$$

(b) Find the general solution for $y'' + 2y' + 10y = 0$.

$$r^2 + 2r + 10 = 0$$
$$r = \frac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$$
$$\boxed{y = c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t}$$

(c) Find the general solution for $y'' - 2y' - 15y = 0$.

$$r^2 - 2r - 15 = 0$$
$$(r-5)(r+3) = 0$$
$$r = 5, -3$$
$$\boxed{y = c_1 e^{5t} + c_2 e^{-3t}}$$

3. [25 pts] Solve the Initial Value Problem $y'' - 4y' + 3y = e^{3t}$, $y(0) = 0$, $y'(0) = -1$.

$r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$

$r = 1, 3$

$y_h = c_1 e^t + c_2 e^{3t}$

$y_p = Ate^{3t}$

$y_p' = Ae^{3t} + 3Ate^{3t}$

$y_p'' = 6Ae^{3t} + 9Ate^{3t}$

$(6A + 9At)e^{3t} - 4(A + 3At)e^{3t} + 3Ate^{3t} \stackrel{?}{=} e^{3t}$

$2A = 1, \quad 0 \cdot A = 0 \implies A = \frac{1}{2}$

$y_p = \frac{1}{2} te^{3t}$

$y = c_1 e^t + c_2 e^{3t} + \frac{1}{2} te^{3t}$

$0 = y(0) = c_1 + c_2 \implies c_1 = -c_2$

$y' = c_1 e^t + 3c_2 e^{3t} + \frac{1}{2} e^{3t} + \frac{3}{2} te^{3t}$

$-1 = y'(0) = c_1 + 3c_2 + \frac{1}{2}$

$= 2c_2 + \frac{1}{2} \implies c_2 = -\frac{3}{4},$

$c_1 = \frac{3}{4}$

$$\boxed{y = \frac{3}{4} e^t - \frac{3}{4} e^{3t} + \frac{1}{2} te^{3t}}$$

4. [26 pts] **(a)** Find a particular solution for $y'' + 2y' + 5y = \cos 2t$.

$r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i \implies \cos 2t$ NOT IN y_h

$y_p = A\cos 2t + B\sin 2t$

$y_p' = -2A\sin 2t + 2B\cos 2t$

$y_p'' = -4A\cos 2t - 4B\sin 2t$

$(A+4B)\cos 2t + (B-4A)\sin 2t \stackrel{?}{=} \cos 2t$

$\begin{cases} A+4B = 1 \\ B-4A = 0 \end{cases}$ $\quad B = 4A \quad 17A = 1 \quad$ $A = 1/17$, $B = 4/17$

$$\boxed{y_p = \frac{1}{17}\cos 2t + \frac{4}{17}\sin 2t}$$

(b) **Fact**: $y_1 = t^2$ and $y_2 = t^3$ are linearly independent solutions for the equation $t^2y'' - 4ty' + 6y = 0$. Find a particular solution for $t^2y'' - 4ty' + 6y = \sqrt{t}$.

$y'' - 4t^{-1}y' + 6t^{-2}y = t^{-3/2}$, so $f(t) = t^{-3/2}$

$W(y_1, y_2) = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix} = t^4$

$v_1' = \frac{-t^3 t^{-3/2}}{t^4} = -t^{-5/2} \rightsquigarrow v_1 = \frac{2}{3}t^{-3/2}$

$v_2' = \frac{t^2 t^{-3/2}}{t^4} = t^{-7/2} \rightsquigarrow v_2 = -\frac{2}{5}t^{-5/2}$

$$\boxed{y_p = \frac{2}{3}t^{-3/2}t^2 - \frac{2}{5}t^{-5/2}t^3} \quad \left(= \frac{4}{15}\sqrt{t}\right)$$