Math 5A - Midterm 2 Review Problems - Linear Algebra

Winter 2009

The exam will focus on topics from Section 3.6 and Chapter 5 of the text, although you may need to know additional material from Chapter 3 (covered in 3C) or from Chapter 4 (covered earlier this quarter). Below is an outline of the key topics and sample problems of the type you may be asked on the test. Many are similar to homework problems you have done–just remember that you will be required to show your work and/or justify your answers on the exam.

3.6: Span, Linear (In)dependence, Basis, Dimension.

1. Determine if each list of vectors is linearly dependent or independent. Justify your answers.
   (a) $(1,2),(2,1)$
   (b) $(2,-2),(-2,2)$
   (c) $(1,2,1),(1,3,1),(0,-1,0)$
   (d) $(1,0,0),(1,1,0),(1,1,1)$
   (e) $x+1, x^2+2x, x^2-2$ in the vector space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2.
2. For each part of Problem 1, find a basis for the span of the listed vectors. What is the dimension of the span in each case? Justify your answers.
3. What is the dimension of the subspace $\{(x,y,z,w) \in \mathbb{R}^4 \mid x+y+z=0\}$ of $\mathbb{R}^4$? Find a basis for this subspace.

5.1: Linear Transformations–Definition and Standard Matrix.

4. Which of the following functions are linear transformations? Justify your answers.
   (a) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x+y, x-y, 2x)$.
   (b) $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x+y+z+1, 0)$.
   (c) $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (\cos x, \sin y)$.
   (d) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (z,x,y)$.
   (e) $T: \mathcal{C}^{(2)}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ defined by $T(f(x)) = f''(x) - f(x)$
5. Find the standard matrix of each of the linear transformations from (a)-(d) above.

5.2: Kernel and Image of a Linear Transformation. Rank and Nullity.

6. Find a basis for the kernel of each linear transformation from Problem 4. Find a basis for the image of each linear transformation from Problem 4 (a)-(d). Justify your answers.

7. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $\ker(T)$ is 1-dimensional and $\mathrm{Im}(T)$ is 2-dimensional. (Would it be possible for $\ker(T)$ and $\mathrm{Im}(T)$ to both be 1-dimensional?)

5.3: Eigenvalues, Eigenvectors, Eigenspaces.

8. Find the eigenvalues, and one eigenvector for each eigenvalue, of the following matrices.

   (a)
$$A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$

   (b)
$$B = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

   (c)
$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

9. $\lambda = 2$ is an eigenvalue of the matrix
$$A = \begin{pmatrix} 4 & -12 & -6 \\ 1 & -4 & -3 \\ -1 & 6 & 5 \end{pmatrix}.$$
Find a basis for the eigenspace of A for the eigenvalue $\lambda = 2$.

5.4: Diagonalization (and Diagonalizability) of Matrices.

10. Are the following matrices diagonalizable? Justify your answers. In each case where the matrix is diagonalizable, give the change of coordinate matrix P such that $P^{-1}AP$ is diagonal.

   (a)
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

(b)

$$B = \begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}$$

(c) C is the 3×3 matrix from Problem 9.