Math 5A - Midterm 2 Review Problems - Solutions

Winter 2009

The exam will focus on topics from Section 3.6 and Chapter 5 of the text, although you may need to know additional material from Chapter 3 (covered in 3C) or from Chapter 4 (covered earlier this quarter). Below is an outline of the key topics and sample problems of the type you may be asked on the test. Many are similar to homework problems you have done–just remember that you will be required to show your work and/or justify your answers on the exam.

3.6: Span, Linear (In)dependence, Basis, Dimension.

1. Determine if each list of vectors is linearly dependent or independent. Justify your answers.

   (a) $(1, 2), (2, 1)$
   Solution. We write $c_1(1, 2) + c_2(2, 1) = (0, 0)$ and solve for c_1 and c_2. This yields two equations $c_1 + 2c_2 = 0$ and $2c_1 + c_2 = 0$, and it follows that $c_1 = c_2 = 0$ is the only solution. Hence the two vectors are linearly independent.

   (b) $(2, -2), (-2, 2)$
   Solution. Notice that $(-2, 2) = -(2, -2)$. Since one vector is a scalar multiple of the other, they are linearly dependent.

   (c) $(1, 2, 1), (1, 3, 1), (0, -1, 0)$
   Solution. We write $c_1(1, 2, 1) + c_2(1, 3, 1) + c_3(0, -1, 0) = (0, 0, 0)$ which yields only 2 equations $c_1 + c_2 = 0$ and $2c_1 + 3c_2 - c_3 = 0$ in 3 unknowns. Hence there must be nonzero solutions: for instance, $c_1 = 1, c_2 = -1, c_3 = -1$ is such a solution. This means the vectors are linearly dependent.

   (d) $(1, 0, 0), (1, 1, 0), (1, 1, 1)$
   Solution. We write $c_1(1, 0, 0) + c_2(1, 1, 0) + c_3(1, 1, 1) = (0, 0, 0)$, which yields 3 equations $c_1 + c_2 + c_3 = 0,\ \ c_1 + c_2 = 0$ and $c_3 = 0$. It follows that $c_1 = c_2 = c_3 = 0$ is the only solution, and thus the vectors are linearly independent.

   (e) $x + 1, x^2 + 2x, x^2 - 2$ in the vector space $\mathbb{P}_2$ of polynomials of degree less than or equal to 2.
   Solution. We write $a(x + 1) + b(x^2 + 2x) + c(x^2 - 2) = 0$, and compare the coefficients of each different power of x on each side of the equation. This yields 3 equations (the first comes from looking at the constant terms, the second from the x-terms and the third from the x^2-terms): $a - 2c = 0$, $a + 2b = 0$ and $b + c = 0$. Solving for a, b, c, we see that there is a free variable, and one nonzero solution is given by $a = 2, b = -1, c = 1$. Hence the vectors are linearly dependent.

2. For each part of Problem 1, find a basis for the span of the listed vectors. What is the dimension of the span in each case? Justify your answers.

   Solution. (a) & (d): Since the given vectors were already linearly independent, they form a basis for their span. For the rest, we must find the largest subset of the given vectors that is linearly independent.

   (b): Since the two vectors $(2, -2)$ and $(-2, 2)$ point in opposite directions, their span will consist of all scalar multiples of either one of them. Hence we can pick $(2, -2)$ as our basis vector (any set of 1 nonzero vector is automatically linearly independent).

   (c): As shown above $(1, 2, 1) = (1, 3, 1) + (0, -1, 0)$, so $(1, 2, 1)$ belongs to the span of $(1, 3, 1)$ and $(0, -1, 0)$. Thus the span of all 3 vectors will be the same as the span of just the last 2. Clearly $(1, 3, 1)$ and $(0, -1, 0)$ are linearly independent since neither is a scalar multiple of the other. Thus $\{(1, 3, 1), (0, -1, 0)\}$ is a basis for the span.

   (e): As shown above $x^2 - 2 = -2(x + 1) + (x^2 + 2x)$, so $(x^2 - 2)$ belongs to the span of $(x + 1)$ and $(x^2 + 2x)$. Thus the span of all 3 polynomials is the same as the span of the first 2. The first 2 are also clearly linearly independent, since they are not scalar multiples of each other (one has degree 1 and the other has degree 2). Hence $\{x + 1, x^2 + 2x\}$ is a basis of the span.

3. What is the dimension of the subspace $W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + z = 0\}$ of $\mathbb{R}^4$? Find a basis for this subspace.

   Solution. The subspace is defined as the solution space of the single equation $x + y + z = 0$ in the 4 variables x, y, z, w. Thus there should be 3 free variables, and the subspace will have dimension 3. We choose y, z, w for the free variables–they can take on any values in $\mathbb{R}$–and then the value of x will be determined by $x = -y - z$. Hence we can express the given subspace as

   $$\{(-y-z, y, z, w)|y, z, w \in \mathbb{R}\} = \{y(-1, 1, 0, 0)+z(-1, 0, 1, 0)+w(0, 0, 0, 1)|y, z, w \in \mathbb{R}\},$$

   which equals $span\{(-1, 1, 0, 0), (-1, 0, 1, 0), (0, 0, 0, 1)\}$. Since these 3 vectors are linearly independent, they form a basis for W, and $\dim W = 3$.

5.1: Linear Transformations–Definition and Standard Matrix.

4. Which of the following functions are linear transformations? Justify your answers.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, x - y, 2x)$.
   Solution. Linear. The standard matrix for T has columns equal to $T(e_1) = (1, 1, 2)$ and $T(e_2) = (1, -1, 0)$. Hence, we see that T is linear because it coincides with matrix multiplication:

   $$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + y + z + 1, 0)$.

Solution. Not Linear. Notice that $T(0, 0, 0) = (1, 0)$. However, in order for T to be linear we would need $T(0, 0, 0) = T(0 * (x, y, z)) = 0 * T(x, y, z) = (0, 0)$ by the second axiom.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (\cos x, \sin y)$.

Solution. Not Linear. Again $T(0, 0) = (\cos 0, \sin 0) = (1, 0)$. So T is not linear for the same reason as in (b).

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (z, x, y)$.

Solution. Linear. The standard matrix for T has columns equal to $T(e_1) = (0, 1, 0), T(e_2) = (0, 0, 1)$ and $T(e_3) = (1, 0, 0)$. Hence, we see that T is linear because it coincides with matrix multiplication:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(e) $T : \mathcal{C}^{(2)}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})$ defined by $T(f(x)) = f''(x) - f(x)$

Solution. Linear. We check that the two axioms are satisfied. Let $f(x)$ and $g(x)$ be two functions in $\mathcal{C}^{(2)}(\mathbb{R})$. Then

$$T(f + g) = (f + g)'' - (f + g) = (f'' - f) + (g'' - g) = T(f) + T(g).$$

If $c \in \mathbb{R}$, then $T(cf) = (cf)'' - (cf) = c(f'' - f) = cT(f)$.

5. Find the standard matrix of each of the linear transformations from (a)-(d) above.

Solution. See the solutions above.

5.2: Kernel and Image of a Linear Transformation. Rank and Nullity.

6. Find a basis for the kernel of each linear transformation from Problem 4. Find a basis for the image of each linear transformation from Problem 4 (a)-(d). Justify your answers.

   Solution. (a) To calculate $\ker(T)$, we set $T(x,y) = (0,0,0)$ and get 3 equations: $x+y=0, x-y=0, 2x=0$. The only solution is $x=y=0$, and thus the zero vector $(0,0)$ is the only element of the kernel. (Technically, a basis for the zero subspace is the empty set $\emptyset$, which has no elements.) The image of T is given by the span of the columns of the standard matrix of T (i.e., it is the column space of this matrix). Hence

   $$\mathrm{Im}(T) = span\{(1,1,2),(1,-1,0)\},$$

   and since these two vectors are linearly independent, they are a basis for the image.

   (d) We calculate $\mathrm{Im}(T)$ first. Again it will be the column space of the standard matrix of T. So

   $$\mathrm{Im}(T) = span\{(0,1,0),(0,0,1),(1,0,0)\} = \mathbb{R}^3,$$

   and these three vectors are linearly independent (in fact, they are just the standard basis vectors in a different order) so they form a basis for $im(T)$. The rank-nullity theorem implies that $\dim\ker(T) = \dim\mathbb{R}^3 - \dim\mathrm{Im}(T) = 3-3 = 0$, so $\ker(T)$ must be the zero subspace. (Again, a basis for $\ker(T)$ is the empty set $\emptyset$.)

   (e) Since $ker(T) = \{y \in \mathcal{C}^{(2)}(\mathbb{R}) \mid y'' - y = 0\}$, we must solve the differential equation $y'' - y = 0$. The characteristic equation is $r^2 - 1 = 0$, and the characteristic roots are $r = 1, -1$. Thus the general solution is $y = c_1e^t + c_2e^{-t}$. This means that a basis for the kernel consists of the two linearly independent solutions $f_1(t) = e^t$ and $f_2(t) = e^{-t}$.

7. Give an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $\ker(T)$ is 1-dimensional and $\mathrm{Im}(T)$ is 2-dimensional. (Would it be possible for $\ker(T)$ and $\mathrm{Im}(T)$ to both be 1-dimensional?)

   Solution. We need a 3×3 matrix T of rank 2. The rank-nullity theorem then implies that the kernel of T has dimension $3-2=1$. So we just have to choose 3 vectors in $\mathbb{R}^3$ for the columns of T such that two of the vectors are linearly independent, and the third is contained in the span of the first two. (Recall that the image of T is just the span of the columns of the matrix.) For instance we can choose

   $$T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

   We have $\ker(T) = span\{(0,0,1)\}$ $\mathrm{Im}(T) = span\{(1,0,0),(0,1,0)\}$ (why?).

5.3: Eigenvalues, Eigenvectors, Eigenspaces.

8. Find the eigenvalues, and one eigenvector for each eigenvalue, of the following matrices.

(a)

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$

Solution. The characteristic polynomial is $p(x) = (1-x)(-2-x)+2 = x^2+x = x(x+1)$. Hence the eigenvalues are $x = 0, -1$. For $x = 0$, the eigenvectors are just the nonzero vectors in $\ker A$. Since the second row of A is -2 times the first row, the kernel is defined by a single equation: it consists of all vectors (y, z) with $y + z = 0$. Thus one such eigenvector is $(1, -1)$. For $x = -1$, the eigenvectors are the nonzero vectors in $\ker(A + I)$, which contains all (y, z) such that $2y + z = 0$. Thus one such eigenvector is $(1, -2)$.

(b)

$$B = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

Solution. The characteristic polynomial is $p(z) = (2-z)(2-z)+4 = z^2-4z+8$, which has two complex roots $z = 2 \pm 2i$, and these are the eigenvalues. For $z = 2+2i$, the eigenvectors are just the nonzero vectors in $\ker(B-(2+2i)I)$. We write down this matrix and convert it to RREF:

$$B-(2+2i)I = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \mapsto \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

Hence any (x, y) with $x - iy = 0$ is an eigenvector: for instance $(i, 1)$.
For $z = 2 - 2i$, we have

$$B-(2-2i)I = \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \mapsto \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}.$$

Hence any (x, y) with $x + iy = 0$ is an eigenvector: for instance $(-i, 1)$.

(c)

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution. Using the fact that the matrix C is upper-triangular, we easily compute its characteristic polynomial $p(t) = \det(C - tI) = (1-t)(2-t)(1-t)$ and thus the eigenvalues of C are $t = 1, 2$. For $t = 1$, we convert the matrix $C - I$ to RREF to compute its kernel:

$$C - I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence any (x, y, z) with $y + z = 0$ is an eigenvector: for instance $(1, 0, 0)$.

For $t = 2$, we convert the matrix $C - 2I$ to RREF to compute its kernel:

$$C - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence any (x, y, z) with $x - y = 0$ and $z = 0$ is an eigenvector: for instance $(1, 1, 0)$.

9. $\lambda = 2$ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 4 & -12 & -6 \\ 1 & -4 & -3 \\ -1 & 6 & 5 \end{pmatrix}.$$

Find a basis for the eigenspace of A for the eigenvalue $\lambda = 2$.

Solution. We convert the matrix $A - 2I$ to RREF to compute its kernel:

$$\begin{aligned} A - 2I &= \begin{pmatrix} 2 & -12 & -6 \\ 1 & -6 & -3 \\ -1 & 6 & 3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & -12 & -6 \\ 1 & -6 & -3 \\ 0 & 0 & 0 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 1 & -6 & -3 \\ 1 & -6 & -3 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -6 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence, the eigenspace is

$$\begin{aligned} \ker(A - 2I) &= \{(x, y, z) \in \mathbb{R}^3 \mid x - 6y - 3z = 0\} \\ &= \{(6y + 3z, y, z) \mid y, z \in \mathbb{R}\} \\ &= \{y(6, 1, 0) + z(3, 0, 1) \mid y, z \in \mathbb{R}\} \\ &= \mathit{span}\{(6, 1, 0), (3, 0, 1)\}. \end{aligned}$$

Since the two vectors $(6, 1, 0)$ and $(3, 0, 1)$ are linearly independent and they span the eigenspace, they form a basis for the eigenspace.

5.4: Diagonalization (and Diagonalizability) of Matrices.

10. Are the following matrices diagonalizable? Justify your answers. In each case where the matrix is diagonalizable, give the change of coordinate matrix P such that $P^{-1}AP$ is diagonal.

(a)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Solution. We start by finding the eigenvalues and eigenvectors. The characteristic polynomial is $p(t) = (1-t)(1-t)-1 = t^2-2t = t(t-2)$. Thus the eigenvalues are $t = 0, 2$. Since there are two distinct eigenvalues (and A is 2×2), there is a theorem that says that A is diagonalizable. To find the change of coordinate matrix P, we still need to find a basis of eigenvectors. When $t = 0$, the eigenvectors are the nonzero elements of the $\ker(A)$: $(1, -1)$ is one such vector. When $t = 2$, we can find an eigenvector by solving $(A-2I)\underline{x} = \underline{0}$, which yields the equation $-x + y = 0$. Thus one eigenvector for eigenvalue 2 is $(1, 1)$. Thus

$$P = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(b)

$$B = \begin{pmatrix} 2 & -2 \\ 0 & 2 \end{pmatrix}$$

Solution. The characteristic polynomial is $p(t) = (2-t)^2$. Thus $t = 2$ is the only eigenvalue for B. The eigenspace for the eigenvalue $t = 2$ is $\ker(B - 2I)$, which is spanned by $(1, 0)$. Since the eigenspace is only one-dimensional, it is not possible to find a basis of $\mathbb{R}^2$ consisting of eigenvectors of B. Hence B is not diagonalizable.

(c) C is the 3×3 matrix from Problem 9.

Solution. We already found two linearly independent eigenvectors $(6, 1, 0)$ and $(3, 0, 1)$ for the eigenvalue 2. Thus to show that C is diagonalizable, we only need to find one more eigenvector that is not spanned by these two. We first must find another eigenvalue. The characteristic polynomial is $\det(C - tI) = (4-t)[(-4-t)(5-t)+18]-1[-12(5-t)+36]-1[36+6(-4-t)] = -t^3+5t^2-8t+4 = -(t-2)^2(t-1)$ (factoring it is not too bad since we know that $(t-2)^2$ should be one factor). Thus the other eigenvalue is $t = 1$. If we try to solve $C\underline{x} = \underline{x}$ we get three equations: $3x-12y-6z = 0$; $x-5y-3z = 0$; and $-x+6y+4z = 0$. Adding the last two yields $y + z = 0$, and then the last one becomes $x = 2y$. Thus we get an eigenvector $(2, 1, -1)$ with eigenvalue 1. We now have a basis of eigenvectors of C, and these make up the columns of our change of coordinate matrix P:

$$P = \begin{pmatrix} 6 & 3 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$