Name: Solutions

Perm No.: ______________

Section Time :

Math 5A - Midterm 2

February 18, 2009

Instructions:

- This exam consists of 4 problems worth 10 points each, for a total of 40 possible points. There is also one extra credit question, worth up to 3 points.
- You must show all your work and fully justify your answers in order to recieve full credit. Please indicate your answers clearly by placing a BOX around them. You may leave your answers in unsimplified form. Partial credit will be awarded for work that is relevant and correct.
- No books, calculators or other devices are allowed. You may use one 3" x 5" notecard.
- BOX YOUR FINAL ANSWERS. Write your answers and work on the test itself, in the space alotted. You may attach additional pages if necessary.

No.	Score
1	
2	
3	
4	
Total	
Bonus	

1. Are the following functions linear transformations? If so, calculate the standard matrix. If not, explain why not.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by $T(x, y) = (x, x, x, x)$ for all $x, y \in \mathbb{R}$

$\boxed{\text{Linear.}}$ T is given by multiplying $\begin{pmatrix} x \\ y \end{pmatrix}$ by the matrix $\boxed{\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}}$

columns are $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (2x + xy, xy - 2y)$ for all $x, y \in \mathbb{R}$.

$\boxed{\text{Not Linear.}}$ $T(1,1) = (3,-1)$

but $T(2 \cdot (1,1)) = T(2,2) = (8, 0) \neq 2(3,-1)$.

T does not respect scalar multiplication.

(c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x, y, z) = (6y, z - 5x)$ for all $x, y, z \in \mathbb{R}$.

$\boxed{\text{Linear}}$ T is given by multiplying $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ by the matrix

$$A = \begin{pmatrix} | & | & | \\ T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \\ | & | & | \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 6 & 0 \\ -5 & 0 & 1 \end{pmatrix}}$$

2. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a) Find a basis for $\mathrm{Im}(T)$. Justify your answer.

$\mathrm{Im}(T) = \text{Column Space of } A = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right)$

These 3 vectors are linearly dependent since

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus $\mathrm{Im}(T) = \mathrm{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right)$

These 2 vectors are Linearly independent since neither is a scalar multiple of the other.

Therefore $\boxed{\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right\}}$ is a basis for $\mathrm{Im}(T)$.

(b) Find a basis for $\ker(T)$. Justify your answer.

$$\ker(T) = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{0}\right\} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \underbrace{\begin{matrix} x + 2z = 0 \\ 2x + y = 0 \end{matrix}}_{\text{1 free variable } x}\right\}$$

$$= \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{matrix} z = -\frac{x}{2} \\ y = -2x \end{matrix}, x \in \mathbb{R}\right\} = \left\{\begin{bmatrix} x \\ -2x \\ -x/2 \end{bmatrix} \middle| x \in \mathbb{R}\right\}$$

$$= \left\{x\begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix} \middle| x \in \mathbb{R}\right\} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}\right\}$$

$\Rightarrow \boxed{\begin{bmatrix} 1 \\ -2 \\ -1/2 \end{bmatrix}}$ is a basis for $\ker(T)$

3. Compute the eigenvalues of the following matrices, and determine an eigenvector for each eigenvalue. (Your answers may involve complex numbers.)

$$(a)\ A = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix}; \qquad (b)\ B = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

a) $p(\lambda) = \det \begin{bmatrix} 4-\lambda & 2 \\ -2 & -\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$

$\Rightarrow \boxed{\lambda = 2}$ EIGENVALUES.

EIGENVECTOR $A\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left.\begin{matrix} 4x + 2y = 2x \\ -2x = 2y \end{matrix}\right\} x = -y$

$\Rightarrow \boxed{\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

b) EIGENVALUES $p(\lambda) = \det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 10 = 0$

$\lambda = \frac{6 \pm \sqrt{6^2 - 40}}{2} = \boxed{3 \pm i}$

EIGENVECTORS $B\begin{pmatrix} x \\ y \end{pmatrix} = (3+i)\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 3x - y = (3+i)x \\ x + 3y = (3+i)y \end{matrix}$

$\lambda = 3+i$:

$\Rightarrow y = -ix \quad \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$

$\lambda = 3-i$: $B\begin{pmatrix} x \\ y \end{pmatrix} = (3-i)\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} 3x - y = (3-i)x \\ x + 3y = (3-i)y \end{matrix}$

$\Rightarrow y = ix \quad \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}}$

4. Determine if the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

is diagonalizable. If so, give the change of coordinate matrix P and the diagonal matrix $P^{-1}AP$.

We must see if there is a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

<u>EIGENVALUES</u> $p(\lambda) = \det \begin{pmatrix} 2-\lambda & 3 & 1 \\ 0 & -1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{pmatrix} = -(2-\lambda)^2(\lambda+1) = 0$

$\Rightarrow \lambda = 2, -1.$

<u>EIGENVECTORS</u> $\underline{\lambda=2}$: The eigenspace is

$$\ker(A-2I) = \ker \begin{pmatrix} 0 & 3 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \ker \begin{pmatrix} 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 0 & 1 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ (converting to RREF)

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, y + \tfrac{1}{3}z = 0 \right\}$$ x & z are free variables

$$= \left\{ x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix} \middle| \, x, z \in \mathbb{R} \right\}$$

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 0 \\ -1/3 \\ 1 \end{bmatrix}$ are 2 linearly independent eigenvectors w/ eigenvalue 2.

$\underline{\lambda=-1}$: $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} 2x + 3y + z = -x & \Rightarrow x = -y. \\ -y - z = -y & \Rightarrow y \in \mathbb{R} \text{ (free)} \\ 2z = -z & \Rightarrow z = 0 \end{cases}$

$\Rightarrow \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is an eigenvector w/ eigenvalue -1.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$

$\Rightarrow$ $\boxed{\text{A is diagonalizable}}$

$$\boxed{P = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1/3 & -1 \\ 0 & 1 & 0 \end{pmatrix}}$$

$$\boxed{P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

5. **Extra Credit.** Find a 3×3 matrix A that has the following eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^3$:

- $\lambda_1 = 1$ with eigenvector $\vec{v}_1 = (1, 1, 0)$;
- $\lambda_2 = 2$ with eigenvector $\vec{v}_2 = (2, 2, 1)$; and
- $\lambda_3 = 3$ with eigenvector $\vec{v}_3 = (0, -1, 2)$

Such an A would be diagonalizable:

$D = P^{-1}AP$ with $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

and $P = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}$

Thus $A = PDP^{-1}$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 5 & -4 & -2 \\ -2 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -4 & -2 \\ -4 & 4 & 2 \\ 3 & -3 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -3 & 4 & 2 \\ -6 & 7 & 2 \\ 2 & -2 & 2 \end{pmatrix}}$$