Name: Solutions

Perm No.: ____________

Section Time :

Math 5A - Midterm 1

January 28, 2008

Instructions:

- This exam consists of 4 problems worth 10 points each, for a total of 40 possible points.
- You must show all your work and fully justify your answers in order to recieve full credit. Please indicate your answers clearly by placing a BOX around them. You may leave your answers in unsimplified form. Partial credit will be awarded for work that is relevant and correct.
- No books, calculators or other devices are allowed. You may use one 3" x 5" notecard.
- BOX YOUR FINAL ANSWERS. Write your answers and work on the test itself, in the space alotted. You may attach additional pages if necessary.

No.	Score
1	
2	
3	
4	
Total	

1. Solve the initial value problem

$$y'' + 4y' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -6.$$

$$r^2 + 4r + 8 = 0$$

$$r = \frac{-4 \pm \sqrt{16-32}}{2} = -2 \pm 2i$$

$$\Rightarrow \quad y = e^{-2t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$$

$$y(0) = \underbrace{e^{0}}_{1}\left(c_1 \underbrace{\cos 0}_{1} + c_2 \underbrace{\sin 0}_{0}\right)$$

$$= c_1 = 2$$

$$y' = -2e^{-2t}\left(c_1 \cos(2t) + c_2 \sin(2t)\right)$$
$$+ e^{-2t}\left(-2c_1 \sin(2t) + 2c_2 \cos(2t)\right)$$

$$y'(0) = -2c_1 + 2c_2 = -6$$

$$\Rightarrow \quad c_2 = \frac{-6 + 2c_1}{2} = -3 + c_1 = -1$$

$$\boxed{y = e^{-2t}\left(2\cos(2t) - \sin(2t)\right)}$$

2. (a) Find the general solution of the homogeneous equation $y'' - 4y' + 4y = 0$.

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$ double root

$$\boxed{y = c_1 e^{2t} + c_2 t e^{2t}}$$

(b) Find the general solution of the non-homogeneous equation $y'' - 4y' + 4y = \frac{e^{2t}}{t^2}$.

Variation of Parameters

$y_p = v_1 y_1 + v_2 y_2$ $\quad y_1 = e^{2t} \quad y_2 = t e^{2t}$

$y_1' = 2e^{2t} \quad y_2' = (2t+1)e^{2t}$

$v_1' e^{2t} + v_2' t e^{2t} = 0$

$v_1' 2e^{2t} + v_2' (2t+1) e^{2t} = \frac{e^{2t}}{t^2}$

Subtract $2\times$(first equation) from (2nd equation)

$\Rightarrow v_2' e^{2t} = \frac{e^{2t}}{t^2} \Rightarrow v_2' = t^{-2}$

$v_2 = \int t^{-2}\, dt = -t^{-1} = -\frac{1}{t}$

$v_1' = \frac{-v_2' t e^{2t}}{e^{2t}} = -t^{-2} \cdot t = -\frac{1}{t}$

$\Rightarrow v_1 = \int -\frac{1}{t}\, dt = -\ln|t|$

$\Rightarrow y_p = -\ln|t| \cdot e^{2t} + -\frac{1}{t} t e^{2t}$

$= -\ln|t| e^{2t} - e^{2t}$.

$$\boxed{y = y_h + y_p = c_1 e^{2t} + c_2 t e^{2t} - \ln|t| e^{2t}}$$

(The extra $-e^{2t}$ can be absorbed into $c_1 e^{2t}$)
by changing c_1 to $c_1 - 1$.

3 (a) Find the general solution of the homogeneous equation $y'' - 6y' + 5y = 0$.

$$r^2 - 6r + 5 = 0$$
$$(r-5)(r-1) = 0$$
$$r = 5, 1$$
$$\boxed{y = c_1 e^{5t} + c_2 e^{t}}$$

(b) Find the general solution of the non-homogeneous equation $y'' - 6y' + 5y = 12te^{-t}$

Undetermined Coefficients:

guess: $y_p = (At+B)e^{-t}$

$$y_p' = -(At+B)e^{-t} + Ae^{-t} = (-At+A-B)e^{-t}$$
$$y_p'' = -(-At+A-B)e^{-t} + -Ae^{-t}$$
$$= (At-2A+B)e^{-t}$$

$$y_p'' - 6y_p' + 5y_p = \left[(At-2A+B) - 6(-At+A-B) + 5(At+B)\right]e^{-t}$$
$$= \left[12At - 8A + 12B\right]e^{-t}$$
$$= 12te^{-t}$$

$$\Rightarrow 12A = 12 \qquad \Rightarrow A = \frac{12}{12} = 1$$
$$12B - 8A = 0 \qquad B = \frac{8A}{12} = \frac{2}{3}$$

$$y_p = \left(t + \frac{2}{3}\right)e^{-t}$$

$$\boxed{y = \left(t + \frac{2}{3}\right)e^{-t} + c_1 e^{5t} + c_2 e^{t}}$$

4. Consider a mass-spring system with a 2 kg mass attached to the end of the spring. A force of 19.6 N is required to pull the mass 0.2 m from its equilibrium position. The mass is then released with initial velocity $x'(0) = -0.7 m/s$, and we observe that the mass reaches its maximum speed precisely when it passes back through the equilibrium position.

(a) Show that there cannot be any damping (i.e., $b = 0$)

$$mx'' + bx' + kx = 0.$$

when the mass passes through the equilibrium position, $X = 0$. Since speed is maximum at this instant, X' has a relative extremum (minimum) and thus $X'' = 0$ at this instant.

$\Rightarrow bx' = 0$ since the velocity x' is not 0

$\underline{b = 0.}$

(b) Determine the spring constant k, and find the amplitude A of the motion of the mass. (You may assume there is no damping, even if you cannot do part (a).)

$2x'' + kx = 0.$

$F = kx$

$19.6\text{ N} = k \cdot (.2\text{ m})$

$\Rightarrow \boxed{k = \frac{19.6}{.2} = 98}$

$2x'' + 98x = 0$

$\omega_0 = \sqrt{k/m} = \sqrt{49} = 7.$

$X = C_1 \cos(7t) + C_2 \sin(7t)$

$X(0) = .2$

$= C_1 \cancel{\cos 0}^{1} + C_2 \cancel{\sin 0}^{0}$

$= C_1$

$\Rightarrow \boxed{C_1 = .2}$

$X'(0) = -.7$

$X'(t) = -7C_1 \sin(7t) + 7C_2 \cos(7t)$

$X'(0) = -7C_1 \underbrace{\cancel{\sin 0}}_{0} + 7C_2 \underbrace{\cancel{\cos 0}}_{1}$

$= 7C_2$

$\Rightarrow \boxed{C_2 = \frac{-.7}{7} = -.1}$

$A = \sqrt{C_1^2 + C_2^2}$

$= \sqrt{.04 + .01}$

$= \sqrt{.05}$

$\boxed{= \frac{\sqrt{5}}{10}} \approx .223\text{ m}$