Math 5A - Solutions to Final Exam Review Problems
Winter 2009

Solving 2x2 Homogeneous, Linear Systems of DEs. Details on finding the eigenvalues and eigenvectors have been left out. Of course, on the exam you will be expected to show the work for these calculations.

- **6.2: Real Eigenvalues.**

1. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix} \vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   Solution. The eigenvalues of the matrix A are $\lambda_1 = -1$ and $\lambda_2 = -2$, and the eigenvectors are $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (of course, scalar multiples of these eigenvectors would also be acceptable). Thus the general solution is

   $$\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

   Plugging in $t = 0$ yields $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence $c_1 = 1$ and $c_2 = 0$ and the unique solution satisfying $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is

   $$\vec{x}(t) = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ e^{-t} \end{pmatrix}.$$

2. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

   Solution. The only eigenvalue is $\lambda = 2$ and an eigenvector is $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Since there are not two linearly independent eigenvectors, we must find a vector $\vec{u}$ such that $(A - \lambda I)\vec{u} = \vec{v}$. One such vector is $\vec{u} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The formula now yields the general solution

   $$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} (t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}).$$

   Plugging in $t = 0$ gives $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Hence $c_1 = 0$ and $c_2 = 1$, and the unique solution satisfying the initial condition is

   $$\vec{x}(t) = e^{2t} (t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) = \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix}.$$

- **6.3: Complex Eigenvalues.**

3. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix} \vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   Solution. The eigenvalues are $\lambda = \pm 2i$, whence $\alpha = 0$ and $\beta = 2$. We know the eigenvectors should be of the form $\vec{u} \pm i\vec{v}$, where $\vec{u} + i\vec{v}$ is the eigenvector for $\lambda = +2i$. We find this eigenvector to be $\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, whence $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The general solution is thus

   $$\vec{x}(t) = c_1(\cos(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + c_2(\sin(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cos(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix}).$$

   Plugging in $t = 0$, we get $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and thus $c_1 = 1$ and $c_2 = 0$. Hence the unique solution satisfying the initial condition is

   $$\vec{x}(t) = \cos(2t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \cos(2t) \\ \cos(2t) + \sin(2t) \end{pmatrix}.$$

4. Find the general solution $\vec{x}(t)$ of the system $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$, and the unique solution that satisfies the initial condition $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   Solution. The eigenvalues are $\lambda = 1 \pm i$, whence $\alpha = \beta = 1$. An eigenvector for $\lambda = 1 + i$ is $\begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, whence $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. The general solution is

   $$\vec{x}(t) = c_1 e^t(\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix}) + c_2 e^t(\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix}).$$

   Plugging in $t = 0$ gives $c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, from which we see that $c_1 = 0$ and $c_2 = -1$. Hence the unique solution satisfying the initial condition is

   $$\vec{x}(t) = -e^t \begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} = \begin{pmatrix} -e^t \sin t \\ e^t \cos t \end{pmatrix}.$$

Phase Plane Portraits for Homogeneous, Linear Systems. Stability of Equilibria. You will not be asked to draw phase plane portraits on the exam. However, you should still be able to recognize the phase plane portrait of a given system. For instance, you may be asked to match several systems with their phase plane portraits.

- **6.2, 6.4: Phase Planes for Systems with Real Eigenvalues.**

5. Sketch several representative trajectories in the phase plane for the system in Problem 1 above. Draw the separatrix, and give the stability classification of the equilibrium at $(0, 0)$.

Solution. The separatrices are the straight-line trajectories along the eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Since both eigenvalues are negative, these trajectories are approaching the origin. The equilibrium point $(0,0)$ is thus a *stable, attracting node*. The rest of the trajectories will typically start off parallel to the line $\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (heading toward the origin) – since $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is the eigenvector for the eigenvalue -2 of largest absolute value – and then they will curve parallel to the line $\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and continue heading toward the origin.

6. In Problem 2, above, is the equilibrium at $(0,0)$ stable or unstable? Justify.

   Solution. The stability of the equilibrium point at $(0,0)$ is determined by the sign of the eigenvalues of the matrix A. Since the matrix A has only one eigenvalue $\lambda = 2 > 0$, the equilibrium is *unstable* (it would be stable if $\lambda < 0$). Here, $(0,0)$ is called a *degenerate node*.

- **6.3, 6.4: Phase Planes for Systems with Complex Eigenvalues.**

7. For Problems 3 and 4, above, determine the stability of the equilibrium point $(0,0)$, and describe the behavior of the trajectories near the origin.

   Solution. 3) Since the eigenvalues are $\pm 2i$, which are pure imaginary, the equilibrium point $(0,0)$ will be a *center equilibrium*, which is *neutrally stable*. The trajectories will be ellipses centered at the origin.

   4) Since the eigenvalues are $1 \pm i$, with $\alpha = 1 > 0$, the equilibrium point $(0,0)$ will be *unstable*. The trajectories will be repelling spirals.

Solving Nonhomogeneous, Linear Systems (6.7).

8. Find a particular solution to $\vec{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ 10\sin t \end{pmatrix}$.

   Solution. We use the method of undetermined coefficients. Thus we guess

$$\vec{x}_p = \begin{pmatrix} A\cos t + B\sin t \\ C\cos t + D\sin t \end{pmatrix}, \text{ so then} \vec{x}_p' = \begin{pmatrix} -A\sin t + B\cos t \\ -C\sin t + D\cos t \end{pmatrix}.$$

   We now set

$$\vec{x}_p' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\vec{x}_p + \begin{pmatrix} 0 \\ 10\sin t \end{pmatrix} = \begin{pmatrix} (A+4C)\cos t + (B+4D)\sin t \\ (A+C)\cos t + (B+D+10)\sin t \end{pmatrix},$$

   which yields 4 equations in A, B, C, D once we equate the coefficients of $\cos t$ and $\sin t$ on the left and right in each component of the vector. We get

$$\begin{aligned} -A &= B + 4D \\ B &= A + 4C \\ -C &= B + D + 10 \\ D &= A + C, \end{aligned}$$

which we can solve to get $A = 4, B = -8, C = -3, D = 1$. Thus a particular solution is

$$\vec{x}_p(t) = \begin{pmatrix} 4\cos t - 8\sin t \\ -3\cos t + \sin t \end{pmatrix}.$$

9. Find a particular solution to $\vec{x}' = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}$, $t > 0$.

   Solution. We use variation of parameters since the components of the forcing term are not polynomials, exponentials or sin or cos functions. We first have to solve the homogeneous system to find the fundamental matrix F. The eigenvalues of the given matrix are $\lambda_1 = -1$ and $\lambda_2 = 3$, and the eigenvectors are $\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. The columns of the fundamental matrix F will be $e^{\lambda_i} t \vec{v}_i$ for $i = 1, 2$, and thus we get

$$F = \begin{pmatrix} 2e^{-t} & 2e^{3t} \\ -e^{-t} & e^{3t} \end{pmatrix}.$$

   We now guess $\vec{x}_p(t) = F\vec{v}$ where $\vec{v} = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$. The variation of parameters formula (formula (13) on p. 416) says

$$\vec{x}_p(t) = F \int F^{-1} \begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix} dt.$$

   We first calculate

$$F^{-1} = \frac{1}{4e^2 t} \begin{pmatrix} e^{3t} & -2e^{3t} \\ e^{-t} & 2e^{-t} \end{pmatrix} = \begin{pmatrix} e^t/4 & e^t/2 \\ e^{-3t}/4 & e^{-3t}/2 \end{pmatrix}.$$

   Now $vecv = \int F^{-1}\vec{f}\, dt$ gives

$$v_1 = \int (t^{-3} + 2t^{-2}) e^t/4 \, dt, \quad \text{and} \quad v_2 = \int (t^{-3} - 2t^{-2}) e^{-3t}/4 \, dt.$$

   (We will not try to integrate these here. However, on the exam you will be expected to perform the integration at this stage, but the problem will be carefully written so that the integration is actually possible.) Finally, for our particular solution, we have

$$\vec{x}_p(t) = F\vec{v} = \begin{pmatrix} 2e^{-t} \int (t^{-3} + 2t^{-2}) e^t/4 \, dt + 2e^{3t} \int (t^{-3} - 2t^{-2}) e^{-3t}/4 \, dt \\ -e^{-t} \int (t^{-3} + 2t^{-2}) e^t/4 \, dt + e^{3t} \int (t^{-3} - 2t^{-2}) e^{-3t}/4 \, dt \end{pmatrix}.$$

Nonlinear Systems of DE's (7.1-7.2).

10. Find all equilibrium points of the nonlinear system

$$\begin{array}{l} x' = -2x + 3y + xy \\ y' = -x + y - 2xy^2 \end{array},$$

and calculate the linearized system at each. Use the eigenvalues and eigenvectors of the Jacobian matrices to determine whether each equilibrium point is stable/unstable and to describe the behavior of the nearby trajectories. (For fun, you might also try to sketch the phase plane portrait.)

Solution. To find the equilibrium points we set $x' = -2x + 3y + xy = 0$ and $y' = -x + y - 2xy^2 = 0$ and solve for x and y. From the first equation, we get $x = \frac{-3y}{y-2}$. Substituting this in for x in the second equation, and then multiplying by $(y-2)$ gives $y(6y^2 + y + 1) = 0$. The quadratic factor has only complex roots, so the only real solution occurs when $y = 0$, and then also $x = 0$. Thus $(0,0)$ is the only equilibrium point. The linearization at $(0,0)$ is the system $\vec{u}' = J(0,0)\vec{u}$ where

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -2+y & 3+x \\ -1-2y^2 & 1-4xy \end{pmatrix}.$$

Thus $J(0,0) = \left(\begin{smallmatrix} -2 & 3 \\ -1 & 1 \end{smallmatrix}\right)$. The eigenvalues are $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Since $\alpha = -1/2 < 0$, the equilibrium is *stable* and the trajectories spiral toward the origin.

11. Same as 10, but for the system

$$\begin{aligned} x' &= 1 - xy \\ y' &= x - y^3 \end{aligned},$$

Solution. To find the equilibrium points we set $x' = 1 - xy = 0$ and $y' = x - y^3 = 0$ and solve for x and y. From the first equation, we get $x = \frac{1}{y}$. Substituting this in for x in the second equation, and then simplifying yields $y^4 = 1$. Thus $y = \pm 1$ and then $x = 1/y = \pm 1$. The two equilibrium points are $(1,1)$ and $(-1,-1)$.

The linearization at $(1,1)$ is the system $\vec{u}' = J(1,1)\vec{u}$ where

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}.$$

Thus $J(1,1) = \left(\begin{smallmatrix} -1 & -1 \\ 1 & -3 \end{smallmatrix}\right)$. The only eigenvalue is $\lambda = -2$, and an eigenvector is $\vec{v} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. Since $\lambda = -2 < 0$, the equilibrium is *stable* and the trajectories near $(1,1)$ will travel parallel to $\vec{v}$ and then curve around as they approach $(1,1)$. The phase plane portrait near $(1,1)$ should look roughly like a degenerate node (as in problems 2, 6 above).

The linearization at $(-1,-1)$ is the system $\vec{u}' = J(-1,-1)\vec{u}$ where $J(-1,-1) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -3 \end{smallmatrix}\right)$. The eigenvalues are $\lambda_1 = -1 - \sqrt{5} < 0$ and $\lambda_2 = -1 + \sqrt{5} > 0$. The corresponding eigenvectors are $\vec{v}_1 = \left(\begin{smallmatrix} 1 \\ -2+\sqrt{5} \end{smallmatrix}\right)$ and $\vec{v}_2 = \left(\begin{smallmatrix} 1 \\ -2-\sqrt{5} \end{smallmatrix}\right)$. Since the eigenvalues have opposite signs, $(-1,-1)$ is an *unstable saddle node.* The trajectories near $(-1,-1)$ will travel towards $(-1.-1)$ parallel to $\vec{v}_1$ and then curve and travel away from $(-1,-1)$ parallel to $\vec{v}_2$.