Name: Solution

Perm No.: ____________

Math 5A - Final Exam A

March 19, 2009

Instructions:

- This exam consists of 8 problems worth 10 points each, for a total of 80 possible points. Some problems may be a lot more work than others, but each of the 8 problems is weighted equally.
- You must show all your work and fully justify your answers in order to recieve full credit. Please indicate your answers clearly by placing a BOX around them. You may leave your answers in unsimplified form. Partial credit will be awarded for work that is relevant and correct.
- No books, calculators or other devices are allowed. You may use one 3" x 5" notecard.
- BOX YOUR FINAL ANSWERS. Write your answers and work on the test itself, in the space alotted. You may attach additional pages if necessary.

1	
2	
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7	
8	
Total	

$$\sin^2\theta = \tfrac{1}{2}[1-\cos(2\theta)]$$

$$\cos^2\theta = \tfrac{1}{2}[1+\cos(2\theta)]$$

1. Consider the differential equation

$$y'' - 6y' + 9y = 0$$

(a) Find the general solution of the equation.

$$r^2 - 6r + 9 = 0$$
$$(r-3)^2 = 0$$
$$r = 3$$
$$\boxed{y = c_1 e^{3t} + c_2 t e^{3t}}$$

(b) Find the unique solution $y(t)$ such that $y(0) = 1$ and $y'(0) = 2$.

$$y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1 = 1$$
$$y'(t) = 3c_1 e^{3t} + 3c_2 t e^{3t} + c_2 e^{3t}$$
$$y'(0) = 3c_1 + c_2 = 2$$
$$c_2 = 2 - 3c_1 = -1$$
$$\boxed{y(t) = e^{3t} - t e^{3t}}$$

2. Consider the nonhomogeneous second order differential equation

$$y'' + 4y' = \sin(2t).$$

(a) Find the general solution $y_h(t)$ of the homogeneous equation $y'' + 4y' = 0$.

$$r^2 + 4r = 0$$
$$r(r+4) = 0$$
$$r = 0, -4$$
$$y = c_1 e^{0t} + c_2 e^{-4t}$$
$$\boxed{y(t) = c_1 + c_2 e^{-4t}}$$

(b) Find a particular solution $y_p(t)$ of the nonhomogeneous equation.

guess $y_p(t) = A\cos(2t) + B\sin(2t)$

$$y_p'(t) = -2A\sin(2t) + 2B\cos(2t)$$
$$y_p''(t) = -4A\cos(2t) - 4B\sin(2t)$$

$$y_p'' + 4y_p' = (-4A + 8B)\cos(2t) + (-4B - 8A)\sin(2t)$$
$$= \sin(2t)$$

$$\Rightarrow \begin{cases} -4A + 8B = 0 & \Rightarrow A = 2B \\ -4B - 8A = 1 & \Rightarrow 20B = -1 \Rightarrow B = -\frac{1}{20} \\ & \quad A = -\frac{1}{10} \end{cases}$$

$$\Rightarrow \boxed{y_p(t) = -\frac{1}{10}\cos(2t) - \frac{1}{20}\sin(2t)}$$

3. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x + 2y + 3z,\ 3x + 6y + 9z), \quad x, y, z \in \mathbb{R}.$$

(a) What is the standard matrix of T?

$$\left.\begin{aligned} T(\vec{e}_1) &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ T(\vec{e}_2) &= \begin{pmatrix} 2 \\ 6 \end{pmatrix} \\ T(\vec{e}_3) &= \begin{pmatrix} 3 \\ 9 \end{pmatrix} \end{aligned}\right\} \text{columns of standard matrix} \qquad \boxed{A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}}$$

(b) Find a basis for Im(T). Justify your answer.

$$\begin{aligned} \operatorname{Im}(T) = \text{column space of } A &= \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix}\right\} \\ &= \operatorname{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} \end{aligned}$$

since $\begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} 3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Thus $\boxed{\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}}$ is a basis for $\operatorname{Im}(T)$

(c) Find a basis for ker(T). Justify your answer.

$$\ker T = \ker\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{pmatrix}}_{\text{RREF}} = \ker\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \,\middle|\, x + 2y + 3z = 0\right\}$$

y, z = free variables

$$= \left\{\begin{pmatrix} -2y - 3z \\ y \\ z \end{pmatrix} \,\middle|\, y, z \in \mathbb{R}\right\}$$

$$= \left\{y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \,\middle|\, y, z \in \mathbb{R}\right\}$$

$$= \operatorname{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Since $\boxed{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \text{ \& } \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$ are Linearly independent they are a basis for $\ker(T)$.

4. Consider the system of linear differential equations:

$$x'(t) = -2x(t) + 7y(t)$$
$$y'(t) = -2y(t)$$

(a) Find the general solution to the system.

Let $\vec{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, so $\vec{x}' = \underbrace{\begin{pmatrix} -2 & 7 \\ 0 & -2 \end{pmatrix}}_{A} \vec{x}$

$\lambda = -2$ is the only E-value of A.

E-vectors: $A + 2I = \begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an E-vector

We also need $\vec{u}$ such that $\begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix} \vec{u} = \vec{v}$

$\begin{pmatrix} 0 & 7 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow 7u_2 = 1$

$\Rightarrow \vec{u} = \begin{pmatrix} 0 \\ 1/7 \end{pmatrix}$

general solution $\boxed{\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} t \\ 1/7 \end{pmatrix}}$

(b) Find the unique simultaneous solutions $x(t)$ and $y(t)$ such that $x(0) = 1$ and $y(0) = 1$.

$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1/7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow c_1 = 1, \quad c_2 = 7$

$\Rightarrow \boxed{\vec{x}(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-2t} \begin{pmatrix} 7t \\ 1 \end{pmatrix}}$

or $\boxed{\begin{aligned} x(t) &= e^{-2t} + 7te^{-2t} \\ y(t) &= e^{-2t} \end{aligned}}$

(c) In the xy-phase plane for the system, is the equilibrium point at $(0, 0)$ stable or unstable? What does this tell you about the long-term behavior (ie., as $t \to \infty$) of your solution to (b)?

$(0,0) = \boxed{\text{Stable Equilibrium}}$ since $\lambda = -2 < 0$

$\Rightarrow \boxed{\vec{x}(t) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$ as $t \to \infty$ (trajectories approach the origin)

5 Consider the system of linear differential equations:

$$\vec{x}' = \begin{pmatrix} -2 & 5 \\ -5 & -2 \end{pmatrix} \vec{x}, \quad \text{where } \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of the matrix A in the system.

$$P(\lambda) = \lambda^2 + 4\lambda + 4 + 25 = \lambda^2 + 4\lambda + 29 = 0$$

$$\Rightarrow \lambda = -2 \pm 5i \quad \Rightarrow \quad \alpha = -2, \ \beta = 5$$

$$A - (-2+5i)I = \begin{pmatrix} -5i & 5 \\ -5 & -5i \end{pmatrix} \Rightarrow -5ix + 5y = 0 \Rightarrow y = ix$$

so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an E-vector.

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

& $\vec{u} - i\vec{v}$ is the other E-vector.

$$\boxed{\begin{array}{ll} \lambda_1 = -2 + 5i & \lambda_2 = -2 - 5i \\ \vec{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} & \vec{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{array}}$$

(b) Find the general solution of the system.

$$\begin{bmatrix} \vec{x}_{re} \\ \vec{x}_{im} \end{bmatrix} = e^{-2t} \begin{bmatrix} \cos 5t & -\sin 5t \\ \sin 5t & \cos 5t \end{bmatrix} \begin{bmatrix} \vec{u} \\ \vec{v} \end{bmatrix} = \begin{bmatrix} e^{-2t}\left(\cos(5t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(5t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \\ e^{-2t}\left(\sin(5t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(5t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}$$

$$\Rightarrow \vec{x}(t) = c_1 \vec{x}_{re} + c_2 \vec{x}_{im} = \boxed{e^{-2t}\left(c_1 \begin{bmatrix} \cos 5t \\ -\sin 5t \end{bmatrix} + c_2 \begin{bmatrix} \sin(5t) \\ \cos(5t) \end{bmatrix}\right)}$$

(c) What type of equilibrium point is $(0, 0)$? Is it stable or unstable?

$\alpha = -2 < 0 \Rightarrow \boxed{\text{Stable}}$

λ complex $\Rightarrow \boxed{\text{attracting spiral}}$

6. Match the systems below to the correct phase-plane portraits shown on the next page. Justify your answers.

(a) $\vec{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \vec{x}$

Eigenvalues

$p(\lambda) = \lambda^2 - 6\lambda + 8$

$= (\lambda - 4)(\lambda - 2)$

$\Rightarrow \boxed{\lambda = 2, 4}$

$\Rightarrow$ (0,0) = repelling Node

$\Rightarrow$ phase-plane = C or D

Eigenvectors.

$\lambda_1 = 2$ $A - 2I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow x + y = 0$

$\Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$\lambda_2 = 4$ $A - 4I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow -x + y = 0$

$\Rightarrow \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\lambda_2 = 4 > 2 = \lambda_1 \Rightarrow$ trajectories will eventually move parallel to $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\Rightarrow \boxed{D}$

(rather than to $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$)

(b) $\vec{x}' = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix} \vec{x}$.

Eigenvalues $p(\lambda) = \lambda^2 + 2\lambda + 10$

$\Rightarrow \boxed{\lambda = -1 \pm 3i}$

Since the E-values are complex with $\alpha = -1 < 0$

(0,0) is stable & the trajectories will be spirals. $\Rightarrow \boxed{B}$

(c) $\vec{x}' = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \vec{x}$.

EIGENVALUES $p(\lambda) = \lambda^2 - 2\lambda - 8$

$= (\lambda - 4)(\lambda + 2)$

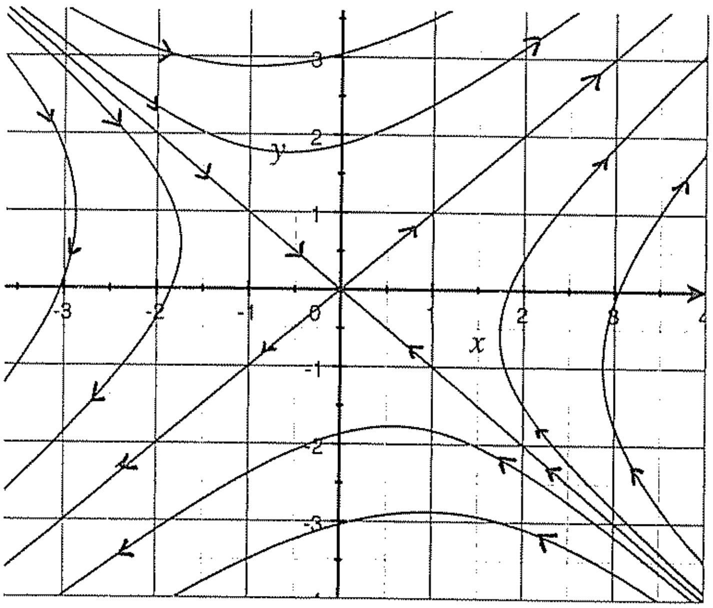
$\Rightarrow \boxed{\lambda = -2, 4}$

Since the eigenvalues have opposite signs

(0,0) = Saddle Node $\Rightarrow \boxed{A}$

6 (continued)

Ⓐ



3
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x
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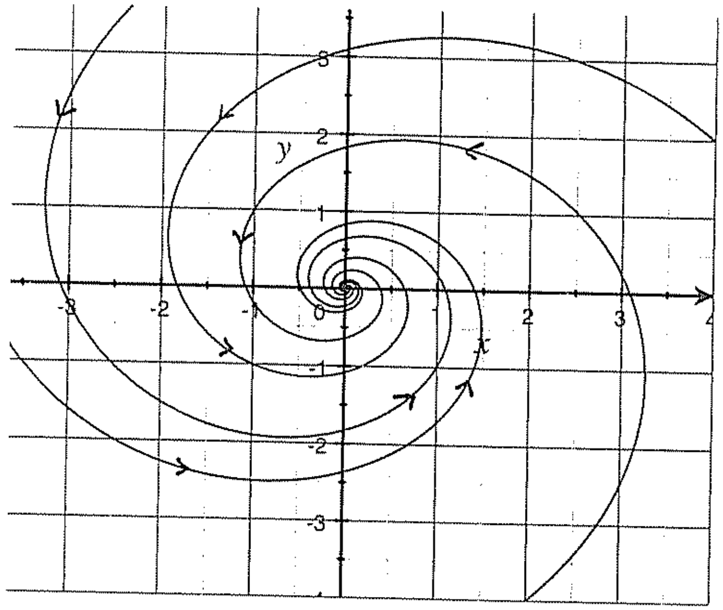

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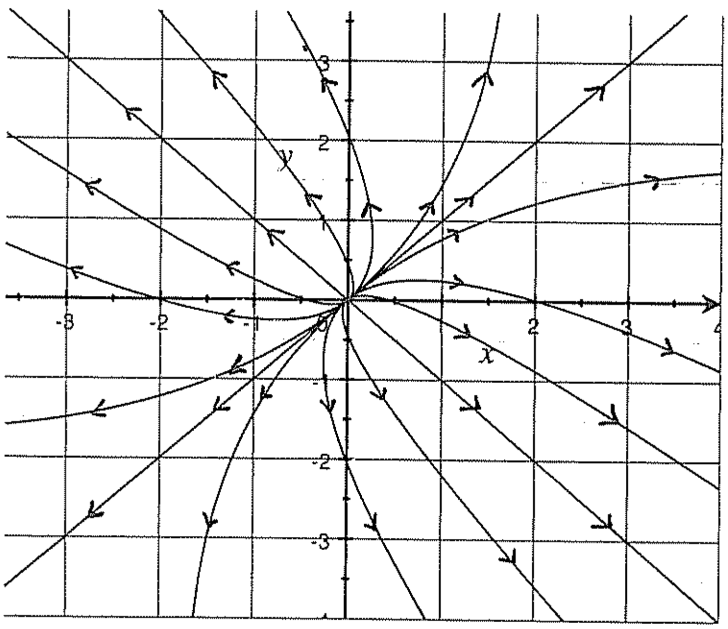

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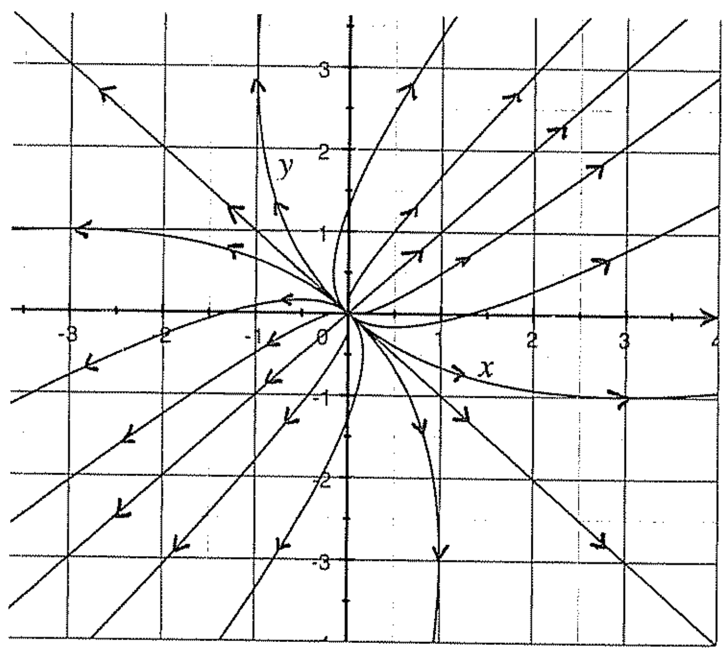

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2
3
4
x
-1
-2
-3

7. Let $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$.

(a) Find the fundamental matrix $F(t)$ for the system $\vec{x}'(t) = A\vec{x}(t)$.

E-values: $p(\lambda) = \lambda^2 - 9 + 8 = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$

$\Rightarrow \lambda = \pm 1.$

E-vectors: $\lambda = 1$: $A - I = \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \Rightarrow 2x - 2y = 0 \Rightarrow x = y \Rightarrow \boxed{\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$\lambda = -1$: $A + I = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix} \Rightarrow 4x - 2y = 0 \Rightarrow y = 2x \Rightarrow \boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

general solution $= \vec{X} = c_1 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} e^t & e^{-t} \\ e^t & 2e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\Rightarrow \boxed{F(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 2e^{-t} \end{pmatrix}}$

(b) Find a particular solution $\vec{x}_p(t)$ to the nonhomogeneous system

$$\vec{x}'(t) = A\vec{x}(t) + \underbrace{\begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}}_{\vec{f}(t)}.$$

$\vec{X}_p(t) = F\int F^{-1}\vec{f}\,dt$

$F^{-1} = \frac{1}{1}\begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^t & e^t \end{bmatrix} \Rightarrow F^{-1}\vec{f} = \begin{bmatrix} 2 - e^{-2t} \\ -e^{2t} + 1 \end{bmatrix}$

$\int F^{-1}\vec{f}\,dt = \begin{bmatrix} \int(2 - e^{-2t})dt \\ \int(-e^{2t}+1)dt \end{bmatrix} = \begin{bmatrix} 2t + \frac{1}{2}e^{-2t} \\ -\frac{1}{2}e^{2t} + t \end{bmatrix}$

$\vec{X}_p = \begin{pmatrix} e^t & e^{-t} \\ e^t & 2e^{-t} \end{pmatrix}\begin{pmatrix} 2t + \frac{1}{2}e^{-2t} \\ t - \frac{1}{2}e^{2t} \end{pmatrix} = \boxed{\begin{pmatrix} 2te^t + \frac{1}{2}e^{-t} + te^{-t} - \frac{1}{2}e^t \\ 2te^t + \frac{1}{2}e^{-t} + 2te^{-t} - e^t \end{pmatrix}}$

$= \cancel{2te^t + te^{-t}}$

8. Consider the nonlinear system of differential equations

$$x' = -x(y^2+1)$$
$$y' = y^2 - 9$$

(a) Find all the equilibrium points of the system.

$x' = -x(y^2+1) = 0 \Rightarrow x = 0$ (since $y^2+1 > 0$)

$y' = y^2 - 9 = 0 \Rightarrow y = \pm 3$.

$\Rightarrow \boxed{(0,3),\ (0,-3)}$

(b) Calculate the Jacobian matrix at each equilibrium point.

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -y^2-1 & -2xy \\ 0 & 2y \end{pmatrix}$$

$$\boxed{J(0,3) = \begin{pmatrix} -10 & 0 \\ 0 & 6 \end{pmatrix}} \qquad \boxed{J(0,-3) = \begin{pmatrix} -10 & 0 \\ 0 & -6 \end{pmatrix}}$$

8. (continued)

(c) Classify each equilibrium point and say whether it is stable or unstable. Justify your answers.

$(0,3)$: $J(0,3)$ has E-values $\lambda = -10, 6$

Opposite signs $\Rightarrow$ $\boxed{(0,3) = \text{unstable, saddle Node}}$

$(0,-3)$: $J(0,-3)$ has E-values $\lambda = -10, -6$

Both negative $\Rightarrow$ $\boxed{(0,-3) = \text{Stable, Attracting Node}}$

(d) Let $\vec{x}(t)$ be the unique solution of the system satisfying the initial condition

$$\vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

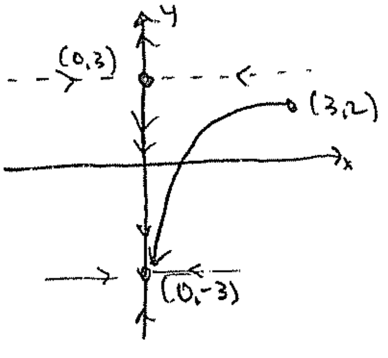
Describe the behavior of $\vec{x}(t)$ (or of the corresponding trajectory in the phase-plane) as $t \to \infty$.

The E-vectors of $J(0,3)$ are $\overset{-10}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ & $\overset{6}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$\Rightarrow$ Separatrix goes along the lines $x = 0$, & $y = 3$

trajectories along $y = 3$ approach $(0,3)$

& trajectories along $x = 0$ move away from $(0,3)$



Since $(0,3)$ is a saddle equilibrium the trajectory starting at $(3,2)$ approaches $(0,3)$, then turns downward paralleling the separatrix $x = 0$.

Since $(0,-3)$ is a stable equilibrium the trajectory will eventually $\boxed{\text{approach } (0,-3)}$