Name: Solution

Perm No.: ____________

Section Time :

Math 3B - Midterm 2

May 17, 2007

Instructions:

- This exam consists of 5 problems worth 6 points each, for a total of 30 possible points. There is also one extra credit question, worth 3 points.
- You must show all your work and fully justify your answers in order to recieve full credit. You may leave your answers in unsimplified form, unless the problem asks you to simplify. However, any functions should be evaluated where possible. This means that you should write "1" instead of "$\ln e$", or "1/2" instead of "$\sin(\pi/6)$". Partial credit will be awarded for work that is relevant and correct.
- You may use one 3" x 5" notecard. No books, calculators or other devices are allowed.
- Write your answers on the test itself, in the space alotted. You may attach additional pages if necessary.

1	
2	
3	
4	
5	
6	
Total	

1. Integrate $\int x \sec^2 x \, dx$.

Integration by Parts

$u = x \qquad dv = \sec^2 x \, dx$

$du = dx \qquad v = \tan x$

$$\int x \sec^2 x \, dx = uv - \int v \, du$$

$$= x \tan x - \int \tan x \, dx$$

$$= x \tan x - \int \frac{\sin x}{\cos x} \, dx$$

let $w = \cos x$

$dw = -\sin x \, dx$

$$= x \tan x + \int \frac{1}{w} \, dw$$

$$= x \tan x + \ln|w| + C$$

$$= \boxed{x \tan x + \ln|\cos x| + C}$$

2. Evaluate $\int_0^{\pi} 4\sin^4 x\, dx$.

$$\int_0^{\pi} 4\sin^4 x\, dx = \int_0^{\pi} 4\left[\tfrac{1}{2}(1-\cos(2x))\right]^2 dx$$

$$= \int_0^{\pi} 1 - 2\cos(2x) + \cos^2(2x) \quad dx$$

$$= \int_0^{\pi} 1 - 2\cos(2x) + \tfrac{1}{2} + \tfrac{1}{2}\cos(4x) \quad dx$$

$$= \left(x - \sin(2x) + \tfrac{1}{2}x + \tfrac{1}{2}\cdot\frac{\sin(4x)}{4}\right)\Big]_0^{\pi}$$

$$= \pi - \sin(2\pi) + \tfrac{1}{2}\pi + \tfrac{1}{8}\sin 4\pi$$
$$- 0 + \sin 0 - 0 - \tfrac{1}{8}\sin 0$$

$$= \boxed{\frac{3\pi}{2}} \qquad (\sin 0 = \sin 2\pi = \sin 4\pi = 0)$$

3. Integrate $\int \frac{x^4+1}{x^2-1}\,dx$.

Long Division:

$$\begin{array}{r|l} & x^2 + 1 \quad \text{Rem } 2. \\ \hline x^2 - 1 & x^4 + 1 \\ & -(x^4 - x^2) \\ \hline & +x^2 + 1 \\ & -(x^2 - 1) \\ \hline & +2 \end{array}$$

$$\Rightarrow \quad \frac{x^4+1}{x^2-1} = x^2 + 1 + \frac{2}{x^2-1}$$

Partial Fractions: $\frac{2}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$

$$\Rightarrow \quad 2 = A(x+1) + B(x-1)$$

$$\Rightarrow \quad 2 = (A+B)x + (A-B)$$

so $\left.\begin{array}{l} A + B = 0 \\ A - B = 2 \end{array}\right\} \begin{array}{l} A = -B \\ 2A = 2 \end{array} \Rightarrow \begin{array}{l} A = 1 \\ B = -1. \end{array}$

$$\therefore \quad \frac{x^4+1}{x^2-1} = x^2 + 1 + \frac{1}{x-1} + \frac{-1}{x+1}$$

$$\Rightarrow \quad \int \frac{x^4+1}{x^2-1}\,dx = \int x^2 + 1 + \frac{1}{x-1} + \frac{-1}{x+1}\,dx$$

$$= \boxed{\frac{x^3}{3} + x + \ln|x-1| - \ln|x+1| + C}$$

4. Evaluate the integral $\int_0^\infty 2xe^{-3x^2}\,dx$ or show that it diverges.

$$\int_0^\infty 2x\,e^{-3x^2}\,dx = \lim_{t\to\infty} \int_0^t 2x\,e^{-3x^2}\,dx$$

let $u = -3x^2$

$du = -6x\,dx$

$$= \lim_{t\to\infty} \int_{x=0}^{x=t} -\frac{1}{3} e^u\,du$$

$$= \lim_{t\to\infty} -\frac{1}{3} e^u \Big]_{x=0}^{x=t} = \lim_{t\to\infty} -\frac{1}{3} e^{-3x^2} \Big]_0^t$$

$$= \lim_{t\to\infty} -\frac{1}{3} e^{-3t^2} + \frac{1}{3} e^0$$

$$= \lim_{t\to\infty} \left(\frac{1}{3} - \underbrace{\frac{1}{3e^{3t^2}}}_{\to 0} \right) = \boxed{\frac{1}{3}}$$

5. Calculate $\int_0^2 x^3\sqrt{4-x^2}\,dx$.

Substitution

Let $u = 4 - x^2$ notice $x^2 = 4 - u$.

$du = -2x\,dx$

$$\int_0^2 x^3\sqrt{4-x^2}\,dx = \int_0^2 x^2\sqrt{4-x^2}\cdot x\,dx$$

$$= \int_4^0 (4-u)\sqrt{u}\left(-\tfrac{1}{2}\right)du$$

$$= \frac{1}{2}\int_0^4 \left(4u^{1/2} - u^{3/2}\right) du$$

$$= \frac{1}{2}\left[\frac{4u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_0^4$$

$$= \frac{1}{2}\left(\frac{64}{3} - \frac{64}{5}\right) = \boxed{\frac{64}{15}}$$

OR trig. substitution:

let $x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

if $x = 0 = 2\sin\theta$

$0 = \sin\theta$

$\Rightarrow \theta = 0$.

if $x = 2 = 2\sin\theta$

$1 = \sin\theta$

$\Rightarrow \theta = \frac{\pi}{2}$.

$$\Rightarrow \int_0^{\pi/2} 8\sin^3\theta\sqrt{4(1-\sin^2\theta)}\cdot 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 32\sin^3\theta\cos^2\theta\,d\theta$$

$u = \cos\theta$

$du = -\sin\theta\,d\theta$

$$= -\int_1^0 32(1-u^2)u^2\,du$$

$$= 32\int_0^1 u^2 - u^4\,du = 32\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big]_0^1$$

$$= 32\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{64}{15}}$$

6. **Extra Credit.** Does the integral $\int_1^\infty \frac{3\,dx}{(2x-1)(x+1)}$ converge or diverge? Justify your answer.

notice for $x \geq 1$, $2x-1 \geq x$ and $x+1 \geq x$

$$\Rightarrow \frac{3}{(2x-1)(x+1)} \leq \frac{3}{x^2}.$$

$$\int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges}$$

so $\int_1^\infty \frac{3}{(2x-1)(x+1)}\,dx$ $\boxed{\text{converges}}$ by the comparison test.

OR $\frac{3}{(2x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x+1} \Rightarrow 3 = (A+2B)x + A - B.$

$$\Rightarrow \left.\begin{array}{l} A+2B=0 \\ A-B=3 \end{array}\right\} \Rightarrow \begin{array}{l} A=2 \\ B=-1. \end{array}$$

$$= \int_1^\infty \left(\frac{2}{2x-1} - \frac{1}{x+1}\right)dx = \lim_{t\to\infty} \Big(\ln|2x-1| - \ln|x+1|\Big)\Big]_1^t$$

$$= \lim_{t\to\infty} \ln\left|\frac{2x-1}{x+1}\right|\Big]_1^t$$

$$= \lim_{t\to\infty} \ln\left|\frac{2t-1}{t+1}\right| - \ln\frac{1}{2}.$$

$$= \ln 2 - \ln 1/2$$

$$= \boxed{\ln 4} \Rightarrow \boxed{\text{converges}}.$$