Math 34B - Solutions to Practice Final

Winter 2008

1. Let $f(t)$ be the amount of water in the tank after t hours. Since the tank is initially full, we have $f(0) = 48$. We also know that $f'(t) = -2 - 4t$ (the negative signs are needed since water is LEAVING the tank). Hence

$$f(t) = \int (-2 - 4t)\ dt = -2t - 2t^2 + C.$$

To find C, we put $t = 0$ to get $C = f(0) = 48$. Thus $f(t) = 48 - 2t - 2t^2$. When the tank is half empty, $f(t)$ will be 24, so we set $f(t) = 24$ and solve for t:

$$\begin{aligned} 48 - 2t^2 - 2t &= 24 \\ 2t^2 + 2t - 24 &= 0 \\ t^2 + t - 12 &= 0 \\ (t-3)(t+4) &= 0 \end{aligned}$$

Hence $t = 3$ or $t = -4$, but only a positive answer makes sense here, so $t = 3$ hours.

2. a) $\displaystyle\int \cos(\pi x + 2/\pi)\ dx = \frac{1}{\pi}\sin(\pi x + 2/\pi) + C.$

b)

$$\begin{aligned} \frac{d}{dx}(\sin(2x)\cos(3x+4)) &= \frac{d}{dx}(\sin(2x)) * \cos(3x+4) + \sin(2x) * \frac{d}{dx}(\cos(3x+4)) \\ &= 2\cos(2x)\cos(3x+4) + \sin(2x) * (-3\sin(3x+4)) \\ &= 2\cos(2x)\cos(3x+4) - 3\sin(2x)\sin(3x+4). \end{aligned}$$

3. a) By the product rule, $(fg)'(5) = f(5)g'(5) + f'(5)g(5) = 1 * 2 + 6 * (-3) = -16$.

b) Let $A(t)$ denote the area of the rectangle at time t. Of course $A(t) = w(t) * h(t)$ where $w(t)$ and $h(t)$ are the width and height of the rectangle at time t, respectively. Thus, by the product rule, we have

$$A'(t) = w(t)h'(t) + w'(t)h(t).$$

At the current value of t, we know that $w(t) = 6$, $h(t) = 10$, and $w'(t) = h'(t) = 2$. Plugging these numbers into the above equation gives us $A'(t) = 6 * 2 + 10 * 2 = 32\ cm^2/sec$.

4. Since $f''(x) = \sin x$, we can integrate to find $f'(x)$ and then integrate again to find $f(x)$. First, we have $f'(x) = \int \sin x\ dx = -\cos x + C$. Since $f'(0) = 0$, we plug in $x = 0$ to get $f'(0) = 0 = -cos 0 + C = -1 + C$. Thus $C = 1$ and $f'(x) = -\cos x + 1$.

   Now, $f(x) = \int(-\cos x + 1)\ dx = -\sin x + x + D$. Since $f(0) = 1 = -\sin 0 + 0 + D = D$, we have $D = 1$ and thus $f(x) = -\sin x + x + 1$.

5. a) The differential equation is a logistic equation with $M = 1$:

$$y'(t) = ky(t)(1 - y(t)),$$

   where k is some positive constant (we don't yet know what the value of k is.)

   b) The general solution is $y(t) = \frac{M}{Ae^{-kt}+1} = \frac{1}{Ae^{-kt}+1}$.

   c) We start keeping track of time at noon, so that $t = 0$ at noon, and $y(0) = 100/1000 = .1$ (don't forget that y is the FRACTION of the population that has heard the rumor). We also have $y'(0) = 270/1000 = .27$. Plugging these 2 numbers into the differential equation from (a), we can solve for k: $.27 = k(.1)(1 - .1) = .09k$, from which we see that $k = 3$. Next we use our answer to (b) to solve for A: $y(0) = .1 = \frac{1}{Ae^{-3*0}+1} = \frac{1}{A+1}$, from which we get $A = 9$. We now have a complete formula for $y(t)$:

$$y(t) = \frac{1}{9e^{-3t} + 1}.$$

   We set this equal to 1/2 and solve for t. We get $9e^{-3t} + 1 = 2$, and thus $e^{-3t} = 1/9$. We now take the ln of both sides and divide by -3 to get $t = -\ln(1/9)/3 = (\ln 9)/3$.

6. The growth equation $y' = 5y$ has general solution $y(t) = y(0)e^{5t}$. Since we are given $y(0) = 2$, the unique solution is $y(t) = 2e^{5t}$.

7. a) The equation for the tangent plane will be of the form $z = b + f_x(-2,3)x + f_y(-2,3)y$, so we first find the partial derivatives f_x and f_y. Since $f(x, y) = x^3 - y^3$, $f_x = 3x^2$ and $f_y = -3y^2$. Plugging in $(x, y) = (-2, 3)$ gives $f_x(-2, 3) = 12$ and $f_y(-2, 3) = -27$. Thus the tangent plane has equation $z = b + 12x - 27y$, and it passes through the point $(-2, 3, -35)$ since $f(-2, 3) = (-2)^3 - 3^3 = -35$. Thus we can plug in $(-2, 3, -35)$ for (x, y, z) to solve for b. This gives us $-35 = b + 12(-2) - 27(3)$, and so $b = 70$. The equation of the tangent plane is thus

$$z = 70 + 12x - 27y.$$

b) The z-intercept of the tangent plane is just b in the equation above, which is 70. Hence the coordinates of the z-intercept are $(0, 0, 70)$.

c) Since we have the equation of the tangent plane at $(-2, 3)$ from part (a), and the point is $(-2.001, 2.998)$ is close to $(-2, 3)$, the tangent approximation of $(-2.001)^3 - (2.998)^3$ is given by the z-value of the tangent plane when $(x, y) = (-2.001, 2.998)$. Plugging these numbers into the equation of the tangent plane we have $(-2.001)^3 - (2.998)^3 \approx 70 + 12(-2.001) - 27(2.998) = 70 - 24.012 - 27(3 - .002) = 70 - 24.012 - 81 + .054 = -34.058$.

We could have also used the formula $f(x + \Delta x, y + \Delta y) \approx f(x, y) + \Delta x * f_x(x, y) + \Delta y * f_y(x, y)$ for $x = -2$, $y = 3$, $\Delta x = -.001$ and $\Delta y = -.002$ (Be careful here, Δx is NEGATIVE). From (a), we know $f_x(-2, 3) = 12$ and $f_y(-2, 3) = -27$ and $f(-2, 3) = -35$, so plugging everything in gives $f(-2.001, 2.998) \approx -35 + (-.001) * 12 + (-.002)(-27) = -35 - .012 + .054 = -34.058$.

8. We first find the partial derivatives $f_x = 4x - y - 5$ and $f_y = 2y - x - 1$. To find the critical point we set both of these derivatives to 0 and solve for x and y. From $4x - y - 5 = 0$, we get $y = 4x - 5$ and we substitute this into the second equation to get $2(4x - 5) - x - 1 = 0$. Simplifying, we have $7x = 11$, so $x = 11/7$. Now $y = 4(11/7) - 5 = 9/7$.

   The minimum value must occur at $(x, y) = (11/7, 9/7)$ so we plug in those values to $f(x, y)$ to find the minimum value: $f(11/7, 9/7) = 2(11/7)^2 + (9/7)^2 - (11/7)(9/7) - 5(11/7) - 9/7 + 2 = -18/7$.