Math 34B - Midterm Solutions

February 8, 2008

1. Integrate.

   (a) $\int_0^1 (3\sqrt{x} + 12x^2 - 2)\ dx$

   Solution.

$$\begin{aligned} \int_0^1 (3\sqrt{x} + 12x^2 - 2)\ dx &= \left(\frac{3x^{3/2}}{3/2} + 4x^3 - 2x\right)\Big]_0^1 \\ &= \left(2x^{3/2} + 4x^3 - 2x\right)\Big]_0^1 \\ &= (2 + 4 - 2) - 0 \\ &= 4. \end{aligned}$$

   (b) $\int 3e^{-x/2}\ dx$

   Solution.

$$\begin{aligned} \int 3e^{-x/2}\ dx &= 3\frac{e^{-x/2}}{-1/2} + C \\ &= -6e^{-x/2} + C. \end{aligned}$$

2. Let $f(x) = 4\sin\left(\frac{\pi x}{2}\right)$.

   (a) Find the amplitude, period and frequency of the sine wave given by $f(x)$.
   Solution. Amplitude $= 4$. Period $= \frac{2\pi}{\pi/2} = 4$. Frequency $= 1/4$.

   (b) The graph of $y = f(x)$ is given below. Find the area of the shaded region.
   Solution. The area of the shaded region is given by the integral of $f(x)$ over one half of a period, starting at 0:

$$\begin{aligned} \int_0^2 4\sin\left(\frac{\pi x}{2}\right)\ dx &= \left(\frac{-4}{\pi/2}\cos\left(\frac{\pi x}{2}\right)\right)\Big]_0^2 \\ &= \frac{-8}{\pi}\cos(\pi) - \frac{-8}{\pi}\cos(0) \\ &= \frac{8}{\pi} + \frac{8}{\pi} \\ &= \frac{16}{\pi}. \end{aligned}$$

3. Let $f(x) = xe^x$.

(a) Find the first and second derivatives of $f(x)$.

Solution. By the product rule, $f'(x) = (1)e^x + (x)e^x = (1+x)e^x$. Differentiating again, we have $f''(x) = (1)e^x + (1+x)e^x = (2+x)e^x$.

(b) Find all critical points of $f(x)$ and use the second derivative test to determine whether each is a relative minimum or relative maximum.

Solution. To find the critical points we set $f'(x) = (1+x)e^x = 0$. Since e^x is always positive, we may divide both sides of the equation by e^x to get $1+x=0$. Hence $x=-1$ is the only critical point.

Plugging in $x=-1$ to the second derivative, we have $f''(-1) = (2-1)e^{-1} = e^{-1} = 1/e > 0$. By the second derivative test, we conclude that $f(x)$ has a relative minimum at $x=-1$.

4. Business is booming at your lemonade stand. You are currently selling 30 cups of lemonade an hour at the price of \$1 a cup. However, due to inflation, the price for a cup of lemonade is currently increasing at the rate of 5 cents an hour, and your sales are currently decreasing at the rate of 3 cups per hour.

(a) What is your current hourly revenue from selling lemonade?

Solution. Let $H(t)$ be the (instantaneous) hourly revenue t hours from now. Let $C(t)$ be the rate (in cups/hour) at which lemonade is being sold at time t, and let $P(t)$ be the price of a cup of lemonade at time t. In general, we have the equation $H(t) = C(t) * P(t)$ for any value of t. The question asks for

$$H(0) = C(0) * P(0) = (30\ \frac{cups}{hour})(1\ \frac{\$}{cup}) = 30\ \frac{\$}{hour}.$$

(b) At what rate (in \$/hour) is your hourly revenue changing at this instant?

Solution. We must find $H'(0)$, which equals $C'(0) * P(0) + C(0) * P'(0)$ by the product rule. We are given $C'(0) = -3\ cups/hour$ and $P'(0) = .05\ \$/hour$. Thus

$$H'(0) = -3 * 1 + 30 * .05 = -1.5\ \$/hour.$$

(c) Use (a) and (b) to estimate your hourly revenue in 2 hours. (Use a linear approximation.)

Solution. Your initial hourly revenue is \$ 30, and is decreasing at the rate of 1.5 \$/hour. So we can approximate that in 2 hours, it will have decreased by $(2\ hours)(1.5\ \$/hour) = \$3$ to \$27.

Notice that this is a linear approximation. We are essentially using the tangent line to $H(t)$ at $t=0$ to approximate $H(2)$. This tangent line has equation $y = H'(0)t + H(0) = -1.5t + 30$ by (a) and (b). Now plug in $t=2$ to get $H(2) \approx -1.5(2) + 30 = 27$.