Math 108B - Take-Home Midterm 2 Solutions

1. Let V be a finite-dimensional vector space. We defined the *dual space* of V as the vector space $V^* = \mathcal{L}(V, F)$ of linear functionals on V. We write V^{**} for the dual space of V^*.

   (a) For $v \in V$, define $\varphi_v : V^* \to F$ by $\varphi_v(f) = f(v)$ for all $f \in V^*$. Show that φ_v is a linear map. (Thus $\varphi_v \in V^{**}$.)

   Solution. If $a \in F$ and $f \in V^*$, we have $\varphi_v(af) = af(v) = a \cdot f(v) = a \cdot \varphi_v(f)$. If $f, g \in V^*$, then $\varphi_v(f + g) = (f + g)(v) = f(v) + g(v) = \varphi_v(f) + \varphi_v(g)$. Hence $\varphi_v : V^* \to F$ is a linear map.

   (b) Show that the function $T : V \to V^{**}$, defined by $T(v) = \varphi_v$ for all $v \in V$, is a linear map.

   Solution. First let $a \in F$ and $v \in V$. Then $T(av) = \varphi_{av}$, which is defined by $\varphi_{av}(f) = f(av)$ for all $f \in V^*$. Since f is linear, we have $\varphi_{av}(f) = f(av) = a \cdot f(v) = a \cdot \varphi_v(f)$. Thus $T(av) = \varphi_{av} = a\varphi_v = aT(v)$. Now let $u, v \in V$. Then $T(u + v) = \varphi_{u+v}$, which is defined by $\varphi_{u+v}(f) = f(u + v) = f(u) + f(v) = \varphi_u(f) + \varphi_v(f)$ for all $f \in V^*$. Thus $T(u + v) = \varphi_{u+v} = \varphi_u + \varphi_v = T(u) + T(v)$, and we have shown that T is a linear map.

   (c) Show that T, as in (b), is an isomorphism. (Recall that, in class, we've already shown that V and V^{**} are isomorphic, i.e., they have the same dimension.)

   Solution. It suffices to show that T is injective, since we already know that V and V^{**} have the same dimension. Thus suppose that $T(v) = \varphi_v = 0$ for some $v \in V$. This means that $\varphi_v(f) = f(v) = 0$ for all $f \in V^*$. However, if $v \neq 0$, we can define a linear functional $f \in V^*$ by completing $\{v\}$ to a basis $\{v, w_1, \ldots, w_n\}$ of V and setting $f(v) = 1$ and $f(w_i) = 0$ for all i. Then clearly, $f(v) \neq 0$, which would contradict $\varphi_v = 0$. Hence we must have $v = 0$. This shows that $null(T) = \{0\}$ and hence T is injective.

2. We say that two inner-product spaces V and W are **isometric** if there exists an invertible isometry $T : V \to W$. Prove that two finite-dimensional inner-product spaces V and W are isometric if and only if $\dim V = \dim W$. (Hint: this is similar to Theorem 3.18 in LADR.)

   Solution. The proof of Theorem 3.18 is based on defining a linear map $T : V \to W$ by $T(v_i) = w_i$ where $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ are bases for V and W, and checking that this T is invertible. By our results on isometries, we know that T will be an isometry if and only if T takes an orthonormal basis of V to an orthonormal basis of W. Thus, the only modification we need to make to the proof of Theorem 3.18 is to choose $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ to be orthonormal bases for V and W, which is always possible by Corollary 6.24. Here are all the details:

$\Leftarrow$: Assume $\dim V = \dim W = n$ and let $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$ to be orthonormal bases for V and W, respectively. Define $T : V \to W$ to be the unique linear map such that $T(v_i) = w_i$ for all i. So $T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$ for all $c_i \in F$. As in the proof of Theorem 3.18, we easily see that T is invertible: an inverse $S : W \to V$ is defined to be the unique linear map such that $S(w_i) = v_i$ for all i. Furthermore, since $\{v_1, \ldots, v_n\}$ is an orthonormal basis and so is $\{Tv_1, \ldots, Tv_n\}$, we know that T is an isometry (this is essentially part (e) of Theorem 7.36 in LADR, and I believe we proved it in class). Therefore, V and W are isometric, by definition.

$\Rightarrow$: Assume that V and W are isometric. By definition, they are also isomorphic. Hence $\dim V = \dim W$ follows from Theorem 3.18 of LADR.

Note: As a consequence: if $\dim V = n$ then V is isometric to F^n with the dot product.

3. Describe all normal $n \times n$ matrices over $\mathbb{C}$ that have only one eigenvalue.

   Solution. Suppose A is a normal $n \times n$ matrix over $\mathbb{C}$ that has only one eigenvalue λ. By the spectral theorem $\mathbb{C}^n$ has an orthonormal basis of eigenvectors for A. Equivalently, there is an invertible change-of-basis matrix C such that $C^{-1}AC$ is a diagonal matrix. Furthermore, the entries on the diagonal of $C^{-1}AC$ must be the eigenvalues of A, i.e., λ, and thus $C^{-1}AC = \lambda I_n$. Multiplying both sides by C on the left and C^{-1} on the right, we have $A = C(\lambda I)C^{-1} = \lambda CC^{-1} = \lambda I$. Thus all normal $n \times n$ matrices with only one eigenvalue are scalar multiples of the identity matrix (in any basis!). Clearly the converse is also true: any scalar multiple of the identity matrix commutes with all matrices, and is thus normal.

4. Suppose that $T : V \to V$ is normal. Prove that

$$\text{null}(T^k) = \text{null}(T) \quad \text{and} \quad \text{range}(T^k) = \text{range}(T) \quad \text{for all integers } k \geq 1.$$

   Solution. Notice first that the set-inclusions $null(T) \subseteq null(T^k)$ and $range(T^k) \subseteq range(T)$ hold for any T. Thus we only need to show that $\dim null(T) = \dim null(T^k)$ and $\dim range(T) = \dim range(T^k)$.

   First assume that $F = \mathbb{C}$. Thus, by the spectral theorem V has an orthonormal basis $\{v_1, \ldots, v_n\}$ of eigenvectors of T. Let $\lambda_i \in \mathbb{C}$ be the eigenvalue going with v_i, so that $Tv_i = \lambda_i v_i$ for all i. Then $T^k v_i = \lambda_i^k v_i$, and thus each v_i is also an eigenvector for T^k with eigenvalue λ_i^k. In particular, the multiplicity of 0 as an eigenvalue of T equals the multiplicity of 0 as an eigenvalue for T^k (i.e., the number of different i such that $\lambda_i = 0$ is the same as the number of different i such that $\lambda_i^k = 0$). Since $null(T)$ equals, by definition, the eigenspace of the eigenvalue 0, we see that the multiplicity of 0 as an eigenvalue for T (or for T^k) equals $\dim null(T)$ (or $\dim null(T^k)$). We now have $\dim null(T) = \dim null(T^k)$, and then by the rank-nullity theorem we have

$$\dim range(T) = \dim V - \dim null(T) = \dim V - \dim null(T^k) = \dim range(T^k).$$

Now assume that $F = \mathbb{R}$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of V, and let A be the matrix of T in this basis. Since T is normal, A is a normal matrix, meaning that A commutes with its (conjugate) transpose: $AA^t = A^tA$. The trick is to now consider the linear map $S \in \mathcal{L}(\mathbb{C}^n)$ defined by the matrix A. Since A is the matrix of S in the standard basis of $\mathbb{C}^n$, which is an orthonormal basis with respect to the dot product, and A commutes with its conjugate transpose, we know that S is also a normal operator. Exactly as above, we see that $\dim null(S) = \dim null(S^k)$, and both are equal to the multiplicity of 0 as an eigenvalue of A (or of A^k). In particular, the multiplicity of 0 as an eigenvalue of A equals the multiplicity of 0 as an eigenvalue of A^k. Since A and A^k represent T and T^k (and $0 \in \mathbb{R}$), 0 has the same multiplicity as an eigenvalue of both T and T^k. Hence $\dim null(T) = \dim null(T^k)$, and the rest follows as in the last sentence of the preceding paragraph and the first paragraph.

(Note: in this part of the proof, it is best to think of eigenvalues of A, and their multiplicities, as corresponding to the roots, with multiplicities, of the characteristic polynomial of A. From this perspective, it is clear that the multiplicity of the real eigenvalue 0 is the same over $\mathbb{R}$ or $\mathbb{C}$.)

Alternate Proof. (without using the spectral theorem)

Claim. If T is normal, then $null(T) = null(T^2)$.

Proof of the claim. Obviously, we have $null(T) \subseteq null(T^2)$. Thus let $v \in null(T^2)$. By Proposition 6.46, we know $null(T) = range(T^*)^{\perp}$, and thus $V = null(T) \oplus range(T^*)$. We can write $v = u + w$ for unique $u \in null(T)$ and $w \in range(T^*)$. Then $T(v) = T(u) + T(w) = T(w)$, so to show $v \in null(T)$, it suffices to show that $w \in null(T)$. First note that $T^*T(w) \in null(T)$ since $T(T^*T(w)) = T(T^*T(v)) = T^*T^2(v) = 0$, where we have used that T is normal. Now, using $null(T) = range(T^*)^{\perp}$, we get $\langle T(w), T(w)\rangle = \langle w, T^*T(w)\rangle = 0$ since $w \in range(T^*)$ and $T^*T(w) \in null(T)$. Thus, $T(w) = 0$ as required.

Next, we can show that $null(T) = null(T^k)$ for all $k \geq 2$, as in the proof of Proposition 8.5 (I won't repeat the argument here). Finally, we can use the rank-nullity theorem as in the first proof to get $\dim range(T) = \dim V - \dim null(T) = \dim V - \dim null(T^k) = \dim range(T^k)$, and the equality $range(T) = range(T^k)$ then follows from the trivial inclusion $range(T^k) \subseteq range(T)$.