Math 108B - Home Work # 6 Solutions

1. If A is an $n \times n$ upper-triangular matrix (i.e., $A_{ij} = 0$ for all $i > j$), show that $\det A = \prod_{i=1}^{n} A_{ii}$.

   Solution. As done in class, we can compute the determinant of A by simplifying the wedge product of the columns of A:

$$A_{11}e_1 \wedge (A_{12}e_1 + A_{22}e_2) \wedge \cdots \wedge (A_{1n}e_1 + \cdots + A_{nn}e_n).$$

   We first distribute the wedge-products accross all the sums, and then use the fact that any wedge product containing some e_i twice is 0. This means that the only (possibly) nonzero term we get when we distribute is from taking e_1 from the first factor, and then e_2 from the second factor (since we can't take e_1 a second time), and then e_3 from the third factor and so forth. Thus the above wedge product simplifies to $A_{11} \cdots A_{nn} e_1 \wedge \cdots \wedge e_n$, and by definition $\det A$ is the scalar $A_{11} \cdots A_{nn}$.

2. Let A be a nilpotent $n \times n$ matrix. Show that A is diagonalizable if and only if $A = 0$.

   Solution. Clearly, if $A = 0$, then A is diagonal and hence diagonalizable. Conversely, assume that A is diagonalizable. This means that $A = C^{-1}DC$ for a diagonal matrix D and an invertible matrix C. Thus $D^m = (CAC^{-1})^m = CAC^{-1}CAC^{-1} \cdots CAC^{-1} = CA^mC^{-1} = 0$. However, if the diagonal entries of D are $d_1, \ldots, d_n$, then the diagonal entries of D^m are just $d_1^m, \ldots, d_n^m$. Since $D^m = 0$, $d_i^m = 0$ for all i, and hence $d_i = 0$ for all i. This shows that $D = 0$ and it follows that $A = C^{-1}DC = 0$.

3. This question asks you to find some 3×3 matrices. Your answers will be non-diagonalizable, since they will each have only 2 linearly independent eigenvectors.

   a) Give an example of a 3×3 matrix with only one eigenvalue (over $\mathbb{C}$), but with a 2-dimensional eigenspace. What are the generalized eigenspaces of $\mathbb{C}^3$ for your example?

   b) Give an example of a 3×3 matrix with only two distinct eigenvalues (over $\mathbb{C}$), each of which has a 1-dimensional eigenspace. What are the generalized eigenspaces of $\mathbb{C}^3$ for your example?

   Solution. Our examples will be upper-triangular matrices, since in this case we can see the eigenvalues and their multiplicities directly from the main diagonal. For (a), call the single eigenvalue λ. The matrix must then have λ in all 3 places along the main diagonal. If we leave a 0 in the $(1,2)$-entry, we see that e_1 and e_2 are eigenvectors. We must now fill in the third column so that no additional eigenvectors involving e_3 arise. For instance, take the matrix

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

   Then the eigenspace corresponding to λ is $null(A - \lambda I) = span(e_1, e_2)$ since $(A - \lambda I)(e_3) = e_2$. Since we only have one eigenvalue, there is only one generalized eigenspace.

Since we know that $V = \mathbb{C}^3$ is the direct sum of all the generalized eigenspaces, this one generalized eigenspace must be all of $\mathbb{C}^3$.

For (b), suppose the two eigenvalues are 1 and 2. One of these must occur with multiplicity 2, so we can suppose the entries on the main diagonal are $1, 2$ and 2. By setting the $(1, 2)$-entry to 0, we can make e_1 an eigenvector with eigenvalue 1 and e_2 an eigenvector with eigenvalue 2. As before, if we place a 1 in the $(2, 3)$-entry, the eigenspace of 2 will be only 1-dimensional. Thus our matrix is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The generalized eigenspaces are $\mathbb{C}e_1$ for the eigenvalue 1 and $span(e_2, e_3)$ for the eigenvalue 2.

4. LADR Solutions (p. 188-190)

3. Suppose that $a_0 v + a_1 Tv + \cdots + a_{m-1} T^{m-1} v = 0$. Applying T^{m-1} to this equation and noting that $T^{m-1} v \neq 0$ while $T^m v = 0$, we get $a_0 T^{m-1} v = 0$. It follows that $a_0 = 0$. Now apply T^{m-2} to the equation $a_1 Tv + \cdots + a_{m-1} T^{m-1} v = 0$ to get $a_1 T^{m-1} v = 0$. Hence $a_1 = 0$. Repeating in this manner we see that all the a_i must be 0. Thus the vectors $v, Tv, \ldots, T^{m-1} v$ are linearly independent.

5. Suppose that $(ST)^n = 0$ for some $n \geq 0$. Then $(TS)^{n+1} = TSTS \cdots TSTS = T(ST)^n S = 0$. Hence TS is also nilpotent.

10. We give a counterexample. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, 0)$ for all $x, y \in \mathbb{R}$. Then $null(T) = \mathbb{R}e_1 = range(T)$. Hence we cannot have $\mathbb{R}^2 = null(T) \oplus range(T)$.

11. We know that $\dim V = \dim null(T^n) + \dim range(T^n)$ so it suffices to show that $null(T^n) \cap range(T^n) = \{0\}$ (Theorem 2.18 then implies that $\dim V = \dim(null(T^n) + range(T^n))$ and hence $null(T^n) + range(T^n) = V$). Let $v \in null(T^n) \cap range(T^n)$. This means that $T^n(v) = 0$ and $v = T^n(w)$ for some $w \in V$. Then $T^{2n}(w) = 0$, which implies that $w \in null(T^{2n}) = null(T^n)$ by Proposition 8.6. Thus $v = T^n(w) = 0$.