Math 108B - Home Work # 5 Solutions

1. LADR Problems, p. 159-160:

11. Let T be a normal operator on the complex inner-product space V. By the spectral theorem there is an orthonormal basis $\{e_1, \ldots, e_n\}$ of V consisting of eigenvectors for T. If $Te_i = \lambda_i e_i$ for $\lambda_i \in \mathbb{C}$, we can define an operator $S \in \mathcal{L}(V)$ by setting $S(e_i) = \sqrt{\lambda_i} e_i$ for all i, and extending S linearly (this means $S(\sum_{i=1}^n c_i e_i) = \sum_{i=1}^n c_i \sqrt{\lambda_i} e_i$ for all $c_i \in \mathbb{C}$). Then we have $S^2(e_i) = S(\sqrt{\lambda_i} e_i) = \lambda_i e_i = Te_i$ for all i, and hence $S^2 = T$.

(Note: for a complex number λ, $\sqrt{\lambda}$ is not well-defined. There are two square roots of λ, and one is always -1 times the other, but in general neither is positive (since they will be complex, not real). However, for this problem, it does not matter which square root you choose. All we need is for the squares of the eigenvalues of S to equal the eigenvalues of T.)

13. We will prove this by the same strategy as in 11. First find an orthonormal basis $\{e_1, \ldots, e_n\}$ for V with $Te_i = \lambda_i e_i$ for all i. This is possible by the spectral theorems since T is self-adjoint. Now define $S \in \mathcal{L}(V)$ by $S(e_i) = \sqrt[3]{\lambda_i} e_i$ and extend it linearly as in 11. (As in the note after 11, cubed roots of complex numbers are not unique, but we can choose any cubed root of λ_i.) Now $S^3(e_i) = \lambda_i e_i = T(e_i)$ for all i, and hence $S^3 = T$.

15. $\Rightarrow$: Assume that U has a basis $\{e_1, \ldots, e_n\}$ of eigenvectors for T. We can define an inner-product on U by the formula

$$\langle a_1 e_1 + \cdots + a_n e_n, b_1 e_1 + \cdots + b_n e_n \rangle = a_1 \bar{b}_1 + \cdots + a_n \bar{b}_n$$

for all $a_i, b_j \in F$. The proof that this defines an inner-product is essentially the same as in Exercise 1 on Homework 2. Furthermore, notice that the basis $\{e_1, \ldots, e_n\}$ is orthonormal with respect to this inner product. Since the matrix for T in this orthonormal basis is diagonal, and hence symmetric, T is a self-adjoint operator with respect to this inner-product.

$\Leftarrow$: If U has an inner-product, for which T is a self-adjoint operator, then the spectral theorem implies that U has an orthonormal basis of eigenvectors for T. In particular, it has a basis of eigenvectors of T.

22. Since $S \in \mathcal{L}(\mathbb{R}^3)$, S has an eigenvalue $\lambda \in \mathbb{R}$ by Theorem 5.26. Since S is an isometry, $|\lambda| = 1$, which implies that $\lambda = \pm 1$. If $x \in \mathbb{R}^3$ is an eigenvector for λ, then $S^2 x = S(\lambda x) = \lambda^2 x = (\pm 1)^2 x = x$.

2. Recall that if U is a subspace of the inner product space V, we defined the reflection in U to be the linear map $R_U : V \to V$ given by

$$R_U = 2P_U - I_V,$$

where P_U is the orthogonal projection onto U and I_V is the identity map. Show that any self-adjoint isometry $T : V \to V$ is a reflection in some subspace U of V. (Hint: U will turn out to be an eigenspace of T. So what are the possible eigenvalues of T?)

Solution. Assume that $T \in \mathcal{L}(V)$ is a self-adjoint isometry. Since T is self-adjoint, all eigenvalues of T are real, and since T is an isometry, all eigenvalues have absolute value 1. Hence the only possible eigenvalues of T are $+1$ and -1. Also, by the spectral theorem, V has an orthonormal basis $\{e_1, \ldots, e_n\}$ of eigenvectors for T. We may order the basis vectors so that $Te_i = e_i$ for $1 \leq i \leq k$ and $Te_j = -e_j$ for $k+1 \leq j \leq n$ for some k with $0 \leq k \leq n$ (the eigenvalues $+1$ and -1 do not necessarily both occur: if $k = 0$, we have $T = -I$ and if $k = n$, we have $T = I$). We let U be the eigenspace corresponding to the eigenvalue 1, that is

$$U = \{v \in V \mid Tv = v\} = span(e_1, \ldots, e_k).$$

Since $\{e_1, \ldots, e_n\}$ is an orthonormal basis, we also have

$$U^{\perp} = span(e_{k+1}, \ldots, e_n) = \{v \in V \mid Tv = -v\}.$$

We can now show that $T = R_U = 2P_U - I_V$. Let $v \in V$ and write $v = u + w$ for $u \in U$ and $w \in U^{\perp}$. Then $T(u) = u$ and $T(w) = -w$, so $T(v) = u - w$. But, by definition, $P_U(v) = u$. Hence $R_U(v) = 2u - (u + w) = u - w = T(v)$. Hence $T = R_U$.