Math 108B - Home Work # 3 Solutions

1. LADR Problems. 10. We have $||1|| = \sqrt{\int_0^1 1^2\, dx} = 1$, so we can take $e_1 = 1$. Now let

$$u_2 = x - \langle x, e_1\rangle e_1 = x - \langle x, 1\rangle 1 = x - \int_0^1 x dx = x - 1/2.$$

Since $||u_2|| = \sqrt{\int_0^1 (x-1/2)^2\, dx} = \sqrt{1/12}$, we let $e_2 = \sqrt{12}x - \sqrt{12}/2$. Now let

$$\begin{aligned} u_3 &= x^2 - \langle x^2, e_1\rangle e_1 - \langle x^2, e_2\rangle e_2 \\ &= x^2 - \int_0^1 x^2\, dx - \int_0^1 x^2(\sqrt{12}x - \sqrt{12}/2)\, dx(\sqrt{12}x - \sqrt{12}/2) \\ &= x^2 - 1/3 - 12(1/12)(x-1/2) \\ &= x^2 - x + 1/6 \end{aligned}$$

Since $||u_3|| = \sqrt{\int_0^1 (x^2 - x + 1/6)^2\, dx} = \sqrt{\int_0^1 (x^4 - 2x^3 + 4x^2/3 - x/3 + 1/36)\, dx} = \sqrt{1/180}$, let $e_3 = \sqrt{180}u_3 = \sqrt{180}(x^2 - x + 1/6)$. Now $\{e_1, e_2, e_3\}$ is an orthonormal basis.

14. According to corollary 6.27, we need to simply apply the Gram-Schmidt process to any basis for $\mathcal{P}_2(\mathbb{R})$ in which the differentiation operator already has an upper triangular matrix. Since this is the case for the basis $\{1, x, x^2\}$, it will also be true for the orthonormal basis constructed in exercise 10. It is also easy to check directly that differentiation has an upper triangular matrix with respect to the basis $\{e_1, e_2, e_3\}$ found in 10.

15. By Theorem 6.29, $V = U \oplus U^\perp$. If $\{u_1, \ldots, u_m\}$ is a basis for U and $\{v_1, \ldots, v_n\}$ is a basis for $U^\perp$ then it is easy to see that $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ is a basis for V. Hence $\dim V = m + n = \dim U + \dim U^\perp$. Alternatively, use Theorem 2.18.

21. We know that $||u - (1,2,3,4)||$ is minimized for $u = P_U(1,2,3,4)$. In order to calculate $P_U(1,2,3,4)$, we first need to find an orthonormal basis for U. Let $e_1 = (1,1,0,0)/||(1,1,0,0)|| = (1/\sqrt{2}, 1/\sqrt{2}, 0, 0)$. Let $u_2 = (1,1,1,2) - (1,1,1,2) \cdot (1/\sqrt{2}, 1/\sqrt{2}, 0, 0)e_1 = (1,1,1,2) - \sqrt{2}e_1 = (0,0,1,2)$. Then $e_2 = u_2/||u_2|| = (0,0,1/\sqrt{5}, 2/\sqrt{5})$. Now $P_U(1,2,3,4) = (1,2,3,4)\cdot(1/\sqrt{2}, 1/\sqrt{2}, 0, 0)e_1 + (1,2,3,4)\cdot(0,0,1/\sqrt{5}, 2/\sqrt{5})e_2 = 3e_1/\sqrt{2} + 11e_2/\sqrt{5} = (3/2, 3/2, 0, 0) + (0, 0, 11/5, 22/5) = (3/2, 3/2, 11/5, 22/5)$.

2. No, nothing changes. We get the same orthonormal basis for $\mathcal{P}_2(\mathbb{C})$ in this case.

3. If U is a *subset* of an inner product space V (but not necessarily a subspace), we can still define

$$U^\perp = \{v \in V \mid \langle v, u\rangle = 0\ \forall u \in U\}.$$

(a) Prove that $U^\perp = \operatorname{span}(U)^\perp$. (Recall, that $\operatorname{span}(U)$ is the subspace of V consisting of all finite F-linear combinations of vectors in U.)

Solution. Let $v \in span(U)^{\perp}$. Then $\langle u, v\rangle = 0$ for all $u \in U$ since $U \subseteq span(U)$. Hence $v \in U^{\perp}$. Conversely, suppose $v \in U^{\perp}$. If $u \in span(U)$, then $u = \sum_{i=1}^{n} c_i u_i$ for scalars $c_i \in F$ and vectors $u_i \in U$. Hence $\langle u, v\rangle = \langle \sum_{i=1}^{n} c_i u_i, v\rangle = \sum_{i=1}^{n} c_i \langle u_i, v\rangle = 0$. Thus $v \in span(U)^{\perp}$.

(b) Use (a) to prove that $(U^{\perp})^{\perp} = \mathrm{span}(U)$.

Solution. Since $span(U)$ is a subspace of V, we can apply Corollary 6.33 to get $span(U) = (span(U)^{\perp})^{\perp} = (U^{\perp})^{\perp}$, where the second equality follows from (a).

In particular, this exercise implies that if $\{u_1, \ldots, u_m\}$ is a basis for the subspace U, then

$$U^{\perp} = \{v \in V \mid \langle v, u_i\rangle = 0 \ \forall i\ \}.$$