Math 108B - Home Work # 1

Due: Friday, April 11, 2008

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 0 & 3 \end{pmatrix}$$

with respect to the standard bases. Find bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ in which the matrix of T is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

2. The matrix

$$\begin{pmatrix} 4 & -1 \\ 2 & 4 \end{pmatrix}$$

represents a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the basis $\{v_1, v_2\}$ where $v_1 = (1, 1)$ and $v_2 = (-1, 1)$. Find the matrix of T with respect to the basis $\{w_1, w_2\}$ where $w_1 = (1, 2)$ and $w_2 = (0, 1)$.

3. Let $T : V \to W$ be a linear transformation, and let $\{v_1, \ldots, v_n\}$ be a basis for V. Show that T is invertible if and only if $\{Tv_1, \ldots, Tv_n\}$ is a basis for W.

4. The **trace** of an $n \times n$ matrix A is defined as the sum of all the entries on the main diagonal of A. That is,

$$tr(A) = \sum_{i=1}^{n} A_{ii},$$

where A_{ij} denotes the entry of A in the i^{th} row and j^{th} column.

   (a) Show that for any two $n \times n$ matrices A and B, $tr(AB) = tr(BA)$.

   (b) Use (a) to show that if X and Y are similar matrices then $tr(X) = tr(Y)$.

5. Let V be an inner-product space, and let W be a subspace of V. Define the **orthogonal complement** of W by

$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W\}.$$

Show that $W^{\perp}$ is a subspace of V.