Math 108A - Take-Home Midterm Solutions

1. Are the following functions linear maps between F-vector spaces? Justify your answers.

   (a) $F = \mathbb{R}$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x, y) = (y, x)$ for all $x, y \in \mathbb{R}$.
   Solution. Yes, T is a linear map. If $a \in F$ and $x, y \in \mathbb{R}$, we have $T(a(x, y)) = T(ax, ay) = (ay, ax) = a(y, x) = aT(x, y)$. Also, if $x', y' \in \mathbb{R}$, then $T((x, y) + (x', y')) = T(x+x', y+y') = (y+y', x+x') = (y, x)+(y', x') = T(x, y)+T(x', y')$.

   (b) $F = \mathbb{C}$ and $S : \mathbb{C} \to \mathbb{C}^2$ is defined by $S(x + iy) = (x + ix, y + iy)$ for all $x, y \in \mathbb{R}$.
   Solution. No, S is not a linear map. Notice that $S(i) = S(0 + 1i) = (0, 1 + i)$, and $S(-1) = S(-1 + 0i) = (-1 - i, 0)$. But $-1 = i^2$, so if S was linear, we would have $S(-1) = S(i^2) = iS(i) = i(0, 1 + i) = (0, -1 + i) \neq (-1 - i, 0)$.
   (Note: S would be a linear map if $F = \mathbb{R}$ instead of $\mathbb{C}$.)

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $T(x, y) = (3x - y, 2y - 2x)$.

   (a) Find $Mat(T, \mathcal{E}, \mathcal{E})$ where $\mathcal{E} = \{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$.
   Solution. The columns for $Mat(T, \mathcal{E}, \mathcal{E})$ will be the coordinates of Te_1 and Te_2 in the standard basis. Since $Te_1 = T(1, 0) = (3, -2) = 3e_1 - 2e_2$ and $Te_2 = T(0, 1) = (-1, 2) = -e_1 + 2e_2$, the matrix for T in the standard basis will be
   $$\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}.$$

   (b) Find $Mat(T, \mathcal{B}, \mathcal{B})$ where $\mathcal{B}$ is the basis $\{(1, 2), (-1, 1)\}$ for $\mathbb{R}^2$.
   Solution. The columns for $Mat(T, \mathcal{B}, \mathcal{B})$ will be the coordinates of $T(1, 2)$ and $T(-1, 1)$ in the basis $\mathcal{B}$. Since $T(1, 2) = (1, 2) = 1(1, 2)+0(-1, 1)$ and $T(-1, 1) = (-4, 4) = 0(1, 2) + 4(-1, 1)$, the matrix for T in the basis $\mathcal{B}$ will be
   $$\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

3. Prove that the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ from Problem 2 is invertible and find a formula for its inverse.

   Solution. To prove T is invertible, it suffices to find an inverse $S \in mathcal L(\mathbb{R}^2)$ such that $ST = TS = Id_{\mathbb{R}^2}$. Since $Te_1 = (3, -2)$ and $Te_2 = (-1, 2)$, the requirements that $STe_1 = e_1$ and $STe_2 = e_2$ show that we must have $S(3, -2) = (1, 0)$ and $S(-1, 2) = (0, 1)$. Since $e_1 = (1, 0) = \frac{1}{2}(3, -2) + \frac{1}{2}(-1, 2)$, we must have
   $$Se_1 = \frac{1}{2}S(3, -2) + \frac{1}{2}S(-1, 2) = (1/2, 1/2).$$

Similarly, since $e_2 = (0,1) = \frac{1}{4}(3,-2) + \frac{3}{4}(-1,2)$, we must have

$$Se_2 = \frac{1}{4}S(3,-2) + \frac{3}{4}S(-1,2) = (1/4, 3/4).$$

Thus the matrix for S with respect to the standard basis will be

$$\begin{pmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{pmatrix},$$

and we have $S(x,y) = (x/2+y/4, x/2+3y/4)$ for all $x, y \in \mathbb{R}$. To check that $S = T^{-1}$, we check that $ST(x,y) = (x,y) = TS(x,y)$ for all $x, y \in \mathbb{R}$. (By results we proved in class, it also suffices to check that $Mat(S,\mathcal{E})Mat(T,\mathcal{E}) = I_2$). We have

$$ST(x,y) = S(3x-y, 2y-2x) = ((3x-y)/2+(2y-2x)/4, (3x-y)/2+3(2y-2x)/4) = (x,y),$$

and

$$\begin{aligned} TS(x,y) &= T(x/2+y/4, x/2+3y/4) \\ &= (3(x/2+y/4) - (x/2+3y/4), 2(x/2+3y/4) - 2(x/2+y/4)) \\ &= (x,y). \end{aligned}$$

4. Let U be a subspace of a finite-dimensional vector space V.

   (a) Show that there exists a linear map $T : V \to V$ with $\text{null}(T) = U$.

   Solution. Let $\{u_1, \ldots, u_m\}$ be a basis for U, and extend it to a basis $\{u_1, \ldots, u_m, v_{m+1}, \ldots, v_n\}$ for V. We define $T : V \to V$ to be the unique linear map such that $T(u_i) = 0$ for all i with $1 \le i \le m$ and $T(v_j) = v_j$ for all j with $m+1 \le j \le n$. (It was shown in class that there is always a unique linear map that sends the basis vectors to any vectors of our choosing. On arbitrary linear combinations of the basis vectors, T must be defined by $T(c_1u_1 + \cdots + c_nv_n) = c_1Tu_1 + \cdots + c_nTv_n$.) Clearly $U = span(u_1, \ldots, u_m) \subseteq null(T)$ by construction. But we also see that

$$\begin{aligned} range(T) &= \{T(c_1u_1 + \cdots + c_nv_n) \mid c_i \in F\} \\ &= \{c_{m+1}v_{m+1} + \cdots + c_nv_n \mid c_i \in F\} \\ &= span(v_{m+1}, \ldots, v_n) \end{aligned}$$

   since $v_j = Tv_j$ for all v_j. Thus $\dim range(T) = n - m$. By the rank-nullity theorem,

$$\dim null(T) = \dim V - \dim range(T) = n - (n-m) = m = \dim U.$$

   Therefore, we must have equality $U = null(T)$.

(b) Show that there exists a linear map $S : V \to V$ with range$(S) = U$.

Solution. We keep the same notation for basis vectors as in (a). We define $S : V \to V$ to be the unique linear map such that $S(u_i) = u_i$ for all i with $1 \leq i \leq m$ and $S(v_j) = 0$ for all j with $m + 1 \leq j \leq n$. The same argument as above shows that $range(S) = span(u_1, \ldots, u_m) = U$.

(c) Give examples of such S and T as above when U is the subspace $\mathbb{R}(1, 1, 1)$ of $V = \mathbb{R}^3$.

Solution. In the above notation, we must complete the basis $\{u_1\}$ for U, where $u_1 = (1, 1, 1)$ to a basis of V. So let $v_2 = (0, 1, 0)$ and $v_3 = (0, 0, 1)$. As above, $T : V \to V$ will be defined by $T(u_1) = 0$, $T(v_2) = v_2$ and $T(v_3) = v_3$. Thus

$$T(x, y, z) = T(xu_1 + (y-x)v_2 + (z-x)v_3) = (y-x)v_2 + (z-x)v_3 = (0, y-x, y-x).$$

Meanwhile, S is defined by $S(u_1) = u_1$ and $S(v_2) = S(v_3) = 0$. Thus we have

$$S(x, y, z) = S(xu_1 + (y - x)v_2 + (z - x)v_3) = xSu_1 = xu_1 = (x, x, x).$$

5. (Extra Credit.) Let U be a subspace of V. Show that there exists a linear map $T : V \to V$ with null$(T) = U$ and range$(T) = U$ if and only if $\dim U = \frac{1}{2} \dim V$.

Solution. $\Rightarrow$: Assume null$(T) = U =$ range(T). By the rank-nullity theorem, we have $\dim V = \dim \mathrm{null}(T) + \dim \mathrm{range}(T) = 2 \dim U$. Thus $\dim U = \frac{1}{2} \dim V$.

$\Leftarrow$: Assume that $\dim U = \frac{1}{2} \dim V$. As in the solution to 4, we choose a basis $\{u_1, \ldots, u_m\}$ of U and extend it to a basis $\{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ of V. We define $T : V \to V$ to be the unique linear map such that $T(v_i) = u_i$ and $T(u_i) = 0$ for all i with $1 \leq i \leq m$. As in the solution to 4(a), we see that $range(T) = span(u_1, \ldots, u_m) = U$. Since $U \subseteq null(T)$ and it follows from the rank-nullity theorem that both subspaces have dimension m, we must have $U = null(T)$.