Math 108A - Take-Home Midterm

Due: May 23, 2008 (4pm)

Instructions and Rules:

- You may use your notes and the texts on this exam. In addition, you may cite any result from lecture, homework problems, or the sections of the text we have covered without proof. Just be sure to make clear which result you are using and how you are using it.
- You may not work together or talk to other people about these problems. Charles and I will not answer any questions directly related to the exam, other than for clarification. We will answer questions about material from lecture, past homework problems, etc.
- Each question is worth 10 points total, except for the extra credit question, which is worth 5 points. You must fully justify your answers in order to receive full credit. Partial credit will be given for work that is relevant and correct.
- Your proofs will be graded for clarity and organization, in addition to correctness. Please try to write neatly and, where applicable, in complete sentences.

1. Are the following functions linear maps between F-vector spaces? Justify your answers.

   (a) $F = \mathbb{R}$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(x, y) = (y, x)$ for all $x, y \in \mathbb{R}$.

   (b) $F = \mathbb{C}$ and $S : \mathbb{C} \to \mathbb{C}^2$ is defined by $S(x + iy) = (x + ix, y + iy)$ for all $x, y \in \mathbb{R}$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $T(x, y) = (3x - y, 2y - 2x)$.

   (a) Find $Mat(T, \mathcal{E}, \mathcal{E})$ where $\mathcal{E} = \{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$.

   (b) Find $Mat(T, \mathcal{B}, \mathcal{B})$ where $\mathcal{B}$ is the basis $\{(1, 2), (-1, 1)\}$ for $\mathbb{R}^2$.

3. Prove that the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ from Problem 2 is invertible and find a formula for its inverse.

4. Let U be a subspace of a finite-dimensional vector space V.

   (a) Show that there exists a linear map $T : V \to V$ with $\mathrm{null}(T) = U$.

   (b) Show that there exists a linear map $S : V \to V$ with $\mathrm{range}(S) = U$.

   (c) Give examples of such S and T as above when U is the subspace $\mathbb{R}(1, 1, 1)$ of $V = \mathbb{R}^3$.

5. (Extra Credit.) Let U be a subspace of V. Show that there exists a linear map $T : V \to V$ with $\mathrm{null}(T) = U$ and $\mathrm{range}(T) = U$ if and only if $\dim U = \frac{1}{2} \dim V$.