Math 108A - Home Work # 3 Solutions

1. LADR Problems:

8. Since U is defined by 2 equations in $\mathbb{R}^5$, we can guess that U will be $5-2=3$ dimensional. So we look for 3 linearly independent vectors in U, and then prove that they in fact span U. To find simple vectors in U, notice that we can choose x_2, x_4 and x_5 freely and then $x_1 = 3x_2$ and $x_3 = 7x_4$ will be determined. We thus let one of these 3 numbers equal 1 and the other 2 equal 0, to get

$$u_1 = (3,1,0,0,0),\ u_2 = (0,0,7,1,0),\ u_3 = (0,0,0,0,1) \in U.$$

These vectors are clearly linearly independent since no two of them are nonzero in the same slot. If $x = (x_1, x_2, x_3, x_4, x_5)$ is an arbitrary element of U with $x_1 = 3x_2$ and $x_3 = 7x_4$, then it is easy to see that $x = x_2u_1 + x_4u_2 + x_5u_3 \in span(u_1, u_2, u_3)$. Hence $\{u_1, u_2, u_3\}$ is a basis for U.

Of course, any set of 3 linearly independent vectors in U would also be a valid basis here.

9. True. Let $p_0 = 1, p_1 = x$ and $p_3 = x^3$. For p_2 we cannot take x^2 since this has degree 2, but we can let $p_2 = x^3 + x^2$. Since $x^2 = p_2 - p_3$, it is obvious that $p_0, \dots, p_3$ still span $\mathcal{P}_3(F)$, and thus form a basis since $\mathcal{P}_3(F)$ has dimension 4.

12. Suppose $p_0, \dots, p_m \in \mathcal{P}_m(F)$ are polynomials such that $p_j(2) = 0$ for each j. Assume, by way of contradiction, that $p_0, \dots, p_m$ are linearly independent. Then, since these are $m+1$ linearly independent elements and $\mathcal{P}_m(F)$ has dimension $m+1$, $\{p_0, \dots, p_m\}$ must be a basis. Hence any polynomial $p(x) \in \mathcal{P}_m(F)$ can be written as a linear combination $p(x) = a_0p_0(x) + \cdots + a_mp_m(x)$. But plugging in $x = 2$ would then yield $p(2) = \sum_{i=0}^m a_ip_i(2) = 0$ for any polynomial $p(x)$ of degree $\leq m$. This is a contradiction.

14. Suppose U and W are 5-dimensional subspaces of $\mathbb{R}^9$ with $U \cap W = \{0\}$. Then $\dim U \cap W = 0$, and hence $\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 10$. Since $U + W$ must also be a subspace of $\mathbb{R}^9$, it must have dimension ≤ 9. Hence we would have $10 \leq 9$, a contradiction.

2. Let $v_1, \dots, v_m$ and u be vectors in a vector space V. Show that

$$u \in span(v_1, \dots, v_m) \ \Leftrightarrow span(v_1, \dots, v_m, u) = span(v_1, \dots, v_m).$$

Solution. $\Rightarrow$: Suppose $u \in span(v_1, \dots, v_m)$. Thus there exist scalars $c_1, \dots, c_m \in F$ such that $u = \sum_{i=1}^m c_iv_i$. If $v \in span(v_1, \dots, v_m, u)$, then $v = \sum_{i=1}^m d_iv_i + d_0u$ for scalars $d_0, \dots, d_m \in F$. Substituting the above expression for u, we get $v = \sum_{i=1}^m (d_i + d_0c_i)v_i \in span(v_1, \dots, v_m)$. Hence $span(v_1, \dots, v_m, u) \subseteq span(v_1, \dots, v_m)$, and the reverse inclusion is trivial since any vector that is a linear combination of $v_1, \dots, v_m$ can also be written as a linear combination of $v_1, \dots, v_m$ and u by adding on $0 = 0u$.

$\Leftarrow$: Assume $span(v_1, \dots, v_m, u) = span(v_1, \dots, v_m)$, then $u \in span(v_1, \dots, v_m, u) = span(v_1, \dots, v_m)$.

3. (a) Prove that $\{v_1, \dots, v_m\}$ is a linearly independent set of vectors if and only if any $u \in span(v_1, \dots, v_m)$ can be written uniquely as a linear combination $u = c_1v_1 + \cdots + c_mv_m$ for scalars $c_1, \dots, c_m \in F$.

   Solution. $\Rightarrow$: Suppose $\{v_1, \dots, v_m\}$ is linearly independent and let $u \in span(v_1, \dots, v_m)$. By definition, there exist scalars $c_1, \dots, c_m \in F$ such that $u = \sum_{i=1}^m c_iv_i$. If there exists another set of scalars $d_1, \dots, d_m \in F$ such that we also have $u = \sum_{i=1}^n d_iv_i$, then we can subtract the second expression for u from the first to get $0 = \sum_{i=1}^m (c_i - d_i)v_i$. Since $\{v_1, \dots, v_m\}$ is linearly independent, we must have $c_i - d_i = 0$ for all i. Thus $c_i = d_i$ for all i, and there is only one way to write u as a linear combination of $v_1, \dots, v_m$.

   $\Leftarrow$: Clearly $0 \in span(v_1, \dots, v_m)$ and $0 = 0v_1 + \cdots + 0v_m$. If this is the unique way of writing 0 as a linear combination of $v_1, \dots, v_m$, then these vectors are linearly independent by definition.

   (b) Prove that $\{v_1, \dots, v_m\}$ is a linearly independent set of vectors if and only if

$$span(v_1, \dots, v_m) = Fv_1 \oplus Fv_2 \oplus \cdots \oplus Fv_m.$$

   (Note: by definition $span(v_1, \dots, v_m) = Fv_1 + Fv_2 + \cdots + Fv_m$.)

   Solution. As noted above, we always have $span(v_1, \dots, v_m) = Fv_1 + Fv_2 + \cdots + Fv_m$. By definition, this sum is direct if and only if any vector in $Fv_1 + \cdots + Fv_m$ can be written uniquely as a sum of one vector from each subspace. Thus the sum is direct if and only if any $u \in span(v_1, \dots, v_m)$ can be written as $u = a_1v_1 + \cdots a_mv_m$ for unique scalars $a_1, \dots, a_m \in F$. By (a) this happens if and only if $\{v_1, \dots, v_m\}$ is linearly independent.