Solution

Math 108A - Quiz # 3

May 20, 2009

Consider the linear map $T : F^2 \to F^2$ defined by

$$T(x, y) = (x - y, x - 2y), \quad \text{for } x, y \in F.$$

Let $\mathcal{E}$ be the standard basis for F^2 and let $\mathcal{B} = \{(3,1), (2,1)\}$ be another basis for F^2.

1. Find the standard matrix for T.

$$\boxed{\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}}$$

$$Mat(T;\mathcal{E}) = \begin{pmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

$T(\vec{e}_1) = T(1,0) = (1,1) =$ 1st column.

$T(\vec{e}_2) = T(0,1) = (-1,-2) =$ 2nd column.

2. Find both change of bases matrices between $\mathcal{B}$ and $\mathcal{E}$. That is, find $Mat(I, \mathcal{E}, \mathcal{B})$ and $Mat(I, \mathcal{B}, \mathcal{E})$ – but don't worry about which is which.

$$Mat(I; \mathcal{B}, \mathcal{E}) = \boxed{\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}}$$

Columns = coordinates of $\mathcal{B}$-vectors relative to standard basis $\mathcal{E}$.

$$Mat(I; \mathcal{E}, \mathcal{B}) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \boxed{\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}}$$

or columns = coordinates of $\vec{e}_1$ & $\vec{e}_2$ relative to $\mathcal{B}$.

$\vec{e}_1 = (3,1) - (2,1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{B}} =$ 1st col.

$\vec{e}_2 = -2\cdot(3,1) + 3(2,1) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}_{\mathcal{B}} =$ 2nd col.

3. Find the matrix $Mat(T; \mathcal{B}, \mathcal{B})$ for T relative to the basis $\mathcal{B}$.

EASY WAY: USE THE DEFINITION:

Columns = coordinates of $T(3,1)$ & $T(2,1)$ in basis $\mathcal{B}$.

$T(3,1) = (2,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\mathcal{B}}$, $\quad T(2,1) = (1,0) = (3,1) - (2,1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{B}}$

$$\Rightarrow Mat(T; \mathcal{B}, \mathcal{B}) = \boxed{\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}}$$

Harder Way: use the change of Basis formula:

$$Mat(T; \mathcal{B}, \mathcal{B}) = Mat(I; \mathcal{E}, \mathcal{B})\, Mat(T; \mathcal{E})\, Mat(I; \mathcal{B}, \mathcal{E})$$

$$= \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$