Name: Solutions.

Perm No.: ____________

Math 108A - Midterm Exam

May 8, 2009

Instructions:

- This exam consists of 4 problems worth 10 points each, for a total of 40 points.
- You must show all your work and fully justify your answers in order to receive full credit.
- If your justification involves a result proved in class, you should summarize what that result says and, if necessary, explain how you are using it.
- Partial credit will be given for work that is relevant and correct.
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. Scratch paper is available if you need it. You may want to work out your solutions first on scratch paper, so that you can write them on the test as neatly as possible. You may attach additional pages if necessary.

1	
2	
3	
4	
Total	

1. Find the dimensions of the following subspaces and justify your answers.

(a) $U = \{(x, y, z) \in \mathbb{F}^3 \mid y = -6z\} \subseteq F^3$.

$$U = \{(x, -6z, z) \mid x, z \in F\}$$
$$= \{x(1,0,0) + z(0,-6,1) \mid x, z \in F\}$$
$$= \operatorname{Span}\{\underbrace{(1,0,0), (0,-6,1)}_{\text{linearly independent}}\}$$

$\Rightarrow$ they are a Basis for U

$\Rightarrow$ $\boxed{\dim U = 2}$

(b) $V = \operatorname{span}\{(1,0,0,1), (2,1,1,2), (1,-2,-2,1), (0,1,1,0)\} \subseteq F^4$.

$$(2,1,1,2) = (0,1,1,0) + 2\cdot(1,0,0,1)$$
$$\Rightarrow V = \operatorname{Span}\{(1,0,0,1), (1,-2,-2,1), (0,1,1,0)\}$$
$$(1,-2,-2,1) = (1,0,0,1) - 2(0,1,1,0)$$
$$\Rightarrow V = \operatorname{Span}\{\underbrace{(1,0,0,1), (0,1,1,0)}_{\text{Linearly Independent}}\}.$$

$\Rightarrow$ they are a basis for V

$\Rightarrow$ $\boxed{\dim V = 2}$

2. Let $T : V \to W$ be a linear map.

(a) Complete the definition of **surjective (onto)**

T is surjective if $\forall$ $\vec{w} \in W$ $\exists$ $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

(b) True or False: Every linear map $T : F^3 \to F^2$ is surjective. Justify your answer.

$\boxed{\text{False}}$. The zero-map

$$T(\vec{v}) = \vec{0} \quad \forall \vec{v} \in V$$

is clearly not surjective.

3. The matrix $P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & -1 \end{pmatrix}$ defines a projection $P : F^3 \to F^2$

Find $\ker(P)$ and $\operatorname{im}(P)$.

$$\ker(P) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \;\middle|\; \begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Row-Reduce P:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -3 & -1 \end{pmatrix} \xrightarrow{-2\cdot} \begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & -3 \end{pmatrix} \cdot -1 \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} +$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{matrix} x + 4z = 0 \\ y + 3z = 0 \end{matrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4z \\ -3z \\ z \end{pmatrix}$$

$$\therefore \boxed{\ker(P) = F \cdot \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}}$$

~~By~~ Note $\dim \ker(P) = 1$.

By the Rank-Nullity Theorem,

$$\dim \operatorname{Im}(P) = \dim F^3 - \dim \ker(P) = 3 - 1 = 2.$$

Since $\operatorname{Im}(P) \subseteq F^2$ & $\dim \operatorname{Im}(P) = 2$

$$\boxed{\operatorname{Im}(P) = F^2}$$

4. Let $v_1, \ldots, v_n$ be vectors in a vector space V over F.

(a) Complete the definition of **linear independence**
$\{v_1, \ldots, v_n\}$ are linearly independent if for all scalars $c_1, \ldots, c_n \in F$

$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ implies $c_1 = c_2 = \cdots = c_n = 0$

(b) True or False — Justify your answers. Are the following statements True FOR ALL linear maps $T : V \to W$?

(i) If $\{v_1, \ldots, v_n\}$ are linearly INDEPENDENT vectors in V, then $\{T(v_1), \ldots, T(v_n)\}$ are linearly INDEPENDENT in W.

$\boxed{\text{False}}$. In general, any non-injective map T will yield a counterexample.

For instance, take T to be the 0-map. $T(\vec{v}) = \vec{0} \ \forall \vec{v} \in V$.

(ii) If $\{v_1, \ldots, v_n\}$ are linearly DEPENDENT vectors in V, then $\{T(v_1), \ldots, T(v_n)\}$ are linearly DEPENDENT in W.

$\boxed{\text{TRUE.}}$ If $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}$ and not all $c_i = 0$, then

$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = T(\vec{0})$

$\Rightarrow c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n) = \vec{0}$ by linearity

$\Rightarrow \{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ are Linearly Dependent.