Math 108A - Midterm Review Solutions.

1. $\mathcal{P}_3(F)$ is the vector space of all polynomials of degree ≤ 3 and with coefficients in F.

   (a) Give an example of a subspace of $\mathcal{P}_3(F)$ of dimension 2. Justify why its dimension is 2, but you don't need to justify why it is a subspace.

   (b) Give an example of a subset of $\mathcal{P}_3(F)$ that is not a subspace. Explain why it is not a subspace.

   Solution. (a) $\mathcal{P}_1(F) = span(1, x)$ is a subspace of $\mathcal{P}_3(F)$ of dimension 2. The dimension is 2 because 1 and x are linearly independent polynomials that span the subspace, and hence they are a basis for this subspace.

   (b) Let U be the subset of $\mathcal{P}_3(F)$ consisting of all polynomials of degree 3. It is not a subspace, since it does not contain the 0 polynomial.

2. Find a basis for the subspace

   $$U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = y + z, y = x + w, z + w = 0\} \subseteq \mathbb{R}^4.$$

   Justify your answer.

   Solution.

   $$\begin{aligned} U &= \{(x, y, z, w) \in \mathbb{R}^4 \mid x = y - w, z = -w\} \\ &= \{(y - w, y, -w, w) \mid y, w \in \mathbb{R}\} \\ &= span\{(1, 1, 0, 0), (-1, 0, -1, 1)\}. \end{aligned}$$

   Since the two vectors $(1, 1, 0, 0)$ and $(-1, 0, -1, 1)$ are linearly independent – neither is a scalar multiple of the other – they form a basis for U.

3. (a) Show that the map $T : \mathcal{P}_3(F) \to \mathcal{P}_4(F)$ defined by $T(p(x)) = (x+1)p(x)$ is a linear map.

   (b) Describe $\ker(T)$ and $\text{im}(T)$.

   Solution. (a) Let $p(x), q(x) \in \mathcal{P}_3(F)$ and let $a \in F$. We check that

   $$T(p(x) + aq(x)) = T(p(x)) + aT(q(x)).$$

   $T(p(x)+aq(x)) = (x+1)(p(x)+aq(x)) = (x+1)p(x)+a(x+1)q(x) = T(p(x))+aT(q(x))$. Thus T is a linear map.

   (b) $\ker(T) = \{p(x) \in \mathcal{P}_3(F) \mid (x + 1)p(x) = 0\} = \{0\}$ since $(x + 1)p(x) = 0$ implies $p(x) = 0$.

   $$\begin{aligned} \text{Im}(T) &= \{T(p(x)) \mid p(x) \in \mathcal{P}_3(F)\} \\ &= \{(x + 1)p(x) \mid p(x) \in \mathcal{P}_3(F)\} \\ &= \{q(x) \in \mathcal{P}_4(F) \mid q(-1) = 0\}. \end{aligned}$$

4. True or False (Explain your reasoning): (a) If $\{u_1, u_2\}$ is linearly independent and $\{v_1, v_2\}$ is linearly independent, then $\{u_1, u_2, v_1, v_2\}$ is linearly independent.

   (b) If $\{u_1, u_2\}$ is a spanning set of V and $\{v_1, v_2\}$ is another spanning set of V, then $\{u_1, u_2, v_1, v_2\}$ is also a spanning set of V.

   Solution. (a) False! For a counterexample, let $u_1 = (1, 0), u_2 = (0, 1)$ and $v_1 = (2, 0)$ and $v_2 = (0, 2)$.

   (b) True! Since $\{u_1, u_2\}$ is a spanning set of V, any vector $v \in V$ can be written as a linear combination of u_1 and u_2. Thus any vector $v \in V$ can also be written as a linear combination of u_1 and u_2 and v_1 and v_2 (for instance, the coefficients of v_1 and v_2 can even be 0). Here v_1 and v_2 can be any vectors – they do not need to form a spanning set themselves.

5. Assume that $V = U \oplus W$ for two subspaces U and W of V. Let $\{u_1, \ldots, u_m\}$ be a basis for U and let $\{w_1, \ldots, w_n\}$ be a basis for W. Prove that $\{u_1, \ldots, u_m, w_1, \ldots, w_n\}$ is a basis for V. (Hint: what do you know about $\dim U \oplus W$?)

   Solution. We know that $\dim V = \dim U \oplus W = \dim U + \dim W = m + n$. Thus to show that $\{u_1, \ldots, u_m, w_1, \ldots, w_n\}$ is a basis for V, it suffices to show that these vectors span V (since there are $m + n = \dim V$ of them). Let $v \in V$ and write $v = u + w$ for $u \in U$ and $w \in W$, which we can do since $V = U + W$. Now we can write $u = c_1 u_1 + \cdots + c_m u_m$ and $w = d_1 w_1 + \cdots + d_n w_n$ for scalars $c_i, d_j \in F$, since $\{u_1, \ldots, u_m\}$ is a basis for U and $\{w_1, \ldots, w_n\}$ is a basis for W. Finally,

   $$v = u + w = c_1 u_1 + \cdots + c_m u_m + d_1 w_1 + \cdots + d_n w_n \in span(u_1, \ldots, u_m, w_1, \ldots, w_n),$$

   which shows that these $m + n$ vectors span V.

6. What is the dimension of the subspace

   $$U = \{(x_1, x_2, \ldots, x_n) \in F^n \mid x_1 + 2x_2 + \cdots + nx_n = 0\} \subseteq F^n?$$

   (Hint: can you apply the rank-nullity theorem?)

   Solution. Notice that $U = \ker(T)$ where $T : F^n \to F$ is given by matrix multiplication with the $1 \times n$ matrix $(1\ 2\ 3\ \cdots\ n)$. The image of T is a subspace of F, and clearly it is not the zero-subspace, since T is not the zero-map (for instance, $T(e_1) = 1$ so $1 \in Im(T)$.) Thus $\mathrm{Im}(T) = F$, which has dimension 1. Now, by the rank-nullity theorem,

   $$\dim U = \dim \ker(T) = \dim F^n - \dim \mathrm{Im}(T) = \dim F^n - \dim F = n - 1.$$