Math 108A - Midterm Review

The Midterm Exam will cover the material in the text LADR up through page 47, AND some material from pages 53-58 (in the section Invertibility). It will consist of a couple problems similar to the quiz problems and a couple longer problems similar to the homework. You should KNOW the definitions/meanings of the following terms:

- Vector Space over F.
- Subspace (of a vector space).
- Span (of a set of vectors).
- Linear Dependence, Linear Independence (of a set of vectors).
- The Sum $U_1 + \cdots + U_m$ of subspaces.
- Direct Sum Decomposition $V = U_1 \oplus \cdots \oplus U_m$ of a vector space V into subspaces $U_1, \ldots, U_m$.
- Basis (for a vector space).
- Dimension (of a vector space).
- Linear Map $T : V \to W$.
- Injective/ One-to-One.
- Surjective/ Onto.
- Kernel/ Null Space (of a linear map).
- Image/ Range (of a linear map).
- Isomorphism, Isomorphic Vector Spaces (see p. 55).

You should be familiar with the following Theorems from the book and class: 1.9, **2.6**, 2.8, **2.10**, **2.12**, **2.14**, 2.15, 2.16, 2.17, **2.18**, 3.1, 3.2, 3.3, **3.4**, **3.18**, **3.21**. You will not be tested on the exact statements of these theorems, but an understanding of these basic results is essential for solving the problems.

Practice Problems.

1. $\mathcal{P}_3(F)$ is the vector space of all polynomials of degree ≤ 3 and with coefficients in F.

   (a) Give an example of a subspace of $\mathcal{P}_3(F)$ of dimension 2. Justify why its dimension is 2, but you don't need to justify why it is a subspace.

   (b) Give an example of a subset of $\mathcal{P}_3(F)$ that is not a subspace. Explain why it is not a subspace.

2. Find a basis for the subspace

   $$U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x = y + z, y = x + w, z + w = 0\} \subseteq \mathbb{R}^4.$$

   Justify your answer.

3. (a) Show that the map $T : \mathcal{P}_3(F) \to \mathcal{P}_4(F)$ defined by $T(p(x)) = (x+1)p(x)$ is a linear map.

   (b) Describe $\ker(T)$ and $\mathrm{im}(T)$.

4. True or False (Explain your reasoning): (a) If $\{u_1, u_2\}$ is linearly independent and $\{v_1, v_2\}$ is linearly independent, then $\{u_1, u_2, v_1, v_2\}$ is linearly independent.

   (b) If $\{u_1, u_2\}$ is a spanning set of V and $\{v_1, v_2\}$ is another spanning set of V, then $\{u_1, u_2, v_1, v_2\}$ is also a spanning set of V.

5. Assume that $V = U \oplus W$ for two subspaces U and W of V. Let $\{u_1, \ldots, u_m\}$ be a basis for U and let $\{w_1, \ldots, w_n\}$ be a basis for W. Prove that $\{u_1, \ldots, u_m, w_1, \ldots, w_n\}$ is a basis for V. (Hint: what do you know about $\dim U \oplus W$?)

6. What is the dimension of the subspace

   $$U = \{(x_1, x_2, \ldots, x_n) \in F^n \mid x_1 + 2x_2 + \cdots + nx_n = 0\} \subseteq F^n?$$

   (Hint: can you apply the rank-nullity theorem?)