Math 108A - Home Work # 7 Solutions

1. For the following matrices:

   (a) Find the eigenvalues (over $F = \mathbb{C}$).

   (b) Describe the eigenspace for each eigenvalue. (i.e., describe all the eigenvectors for each eigenvalue).

   (c) Determine whether F^2 or F^3 has a basis consisting of eigenvectors of the matrix.

   (a) and (b)

   - $$A = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$$

     Solution. The characteristic polynomial is $p(x) = (1-x)(-2-x)+2 = x^2+x = x(x+1)$. Hence the eigenvalues are $x = 0, -1$. For $x = 0$, the eigenvectors are just the nonzero vectors in $\ker A$. Since the second row of A is -2 times the first row, the kernel is defined by a single equation: it consists of all vectors (y, z) with $y + z = 0$. Thus the eigenspace V_0 for the eigenvalue 0 is

     $$V_0 = \{(y, z) \in F^2 \mid y + z = 0\} = F(1, -1).$$

     For $x = -1$, the eigenspace is

     $$V_{-1} = \ker(A + I) = \{(y, z) \in F^2 \mid 2y + z = 0\} = F(1, -2).$$

   - $$B = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

     Solution. The characteristic polynomial is $p(z) = (2-z)(2-z)+4 = z^2-4z+8$, which has two complex roots $z = 2 \pm 2i$, and these are the eigenvalues. For $z = 2 + 2i$, the eigenspace is $V_{2+2i} = \ker(B - (2 + 2i)I)$. We write down this matrix and convert it to RREF:

     $$B - (2 + 2i)I = \begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix} \mapsto \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}.$$

     Hence the eigenvectors are precisely those (x, y) with $x - iy = 0$. Hence $V_{2+2i} = F(i, 1)$.

     For $z = 2 - 2i$, we have

     $$B - (2 - 2i)I = \begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix} \mapsto \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}.$$

     Hence any (x, y) with $x + iy = 0$ is an eigenvector, and $V_{2-2i} = F(-i, 1)$.

- $$C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

  Solution. Using the fact that the matrix C is upper-triangular, we easily compute its characteristic polynomial $p(t) = \det(C - tI) = (1-t)(2-t)(1-t)$ and thus the eigenvalues of C are $t = 1, 2$. For $t = 1$, we convert the matrix $C - I$ to RREF to compute its kernel:

  $$C - I = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Hence the eigenspace for $t = 1$ is

  $$V_1 = \{(x, y, z) \in F^3 \mid y + z = 0\} = span\{(1, 0, 0), (0, 1, -1)\}.$$

  For $t = 2$, we convert the matrix $C - 2I$ to RREF to compute its kernel:

  $$C - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

  Hence the eigenspace is

  $$V_2 = \{(x, y, z) \in F^3 \mid x - y = 0 \text{ and } z = 0\} = F(1, 1, 0).$$

(c) For A and B we found two linearly independent eigenvectors, so these form a basis for F^2 in each case. For C we found three linearly independent eigenvectors $(1, 0, 0), (0, 1, -1)$ and $(1, 1, 0)$, and these form a basis for F^3.

2. $\lambda = 2$ is an eigenvalue of the matrix

$$A = \begin{pmatrix} 4 & -12 & -6 \\ 1 & -4 & -3 \\ -1 & 6 & 5 \end{pmatrix}.$$

(a) Find a basis for the eigenspace of A for the eigenvalue $\lambda = 2$.

(b) Does A have any other eigenvalues? If so, find them and find corresponding eigenvectors.

Solution.(a) We convert the matrix $A - 2I$ to RREF to compute its kernel:

$$\begin{aligned} A - 2I &= \begin{pmatrix} 2 & -12 & -6 \\ 1 & -6 & -3 \\ -1 & 6 & 3 \end{pmatrix} \mapsto \begin{pmatrix} 2 & -12 & -6 \\ 1 & -6 & -3 \\ 0 & 0 & 0 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 1 & -6 & -3 \\ 1 & -6 & -3 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -6 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence, the eigenspace is

$$\begin{aligned} \ker(A - 2I) &= \{(x, y, z) \in \mathbb{R}^3 \mid x - 6y - 3z = 0\} \\ &= \{(6y + 3z, y, z) \mid y, z \in \mathbb{R}\} \\ &= \{y(6, 1, 0) + z(3, 0, 1) \mid y, z \in \mathbb{R}\} \\ &= span\{(6, 1, 0), (3, 0, 1)\}. \end{aligned}$$

Since the two vectors $(6, 1, 0)$ and $(3, 0, 1)$ are linearly independent and they span the eigenspace, they form a basis for the eigenspace.

(b) Since $\lambda = 2$ is an eigenvalue of multiplicity 2 and A is a 3×3 matrix, there should be an additional eigenvalue. You could find it by computing the characteristic polynomial of A, but here is a trickier method. First, recognize that A must be diagonalizable: any eigenvector v for the second eigenvalue will be linearly independent from the two eigenvectors we have already found as our basis for the eigenspace V_2. In diagonal form, the matrix will have two 2's on the diagonal and the second eigenvalue λ. Hence its determinant will be 4λ. This is the same as the determinant of A (since $\det(C^{-1}AC) = \det(C)^{-1}\det(A)\det(C) = \det(A)$), which we can easily compute to be 4. Hence $4\lambda = 4$ and $\lambda = 1$. To find the corresponding eigenvectors we must compute the kernel of $A - I$:

$$\begin{aligned} A - I &= \begin{pmatrix} 3 & -12 & -6 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 3 & 3 \\ 1 & -5 & -3 \\ 0 & 1 & 1 \end{pmatrix} \\ &\mapsto \begin{pmatrix} 1 & -5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Hence the eigenvectors are precisely those vectors (x, y, z) such that $x + 2z = 0$ and $y + z = 0$, and the eigenspace is $F(2, 1, -1)$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by the matrix

$$\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 0 & 3 \end{pmatrix}$$

with respect to the standard bases. Find bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ in which the matrix of T is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Solution. For T to have the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with respect to a basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ and a basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$, means simply that $Tu_1 = v_1$ and $Tu_2 = v_2$. Hence $\{u_1, u_2\}$ can remain the standard basis, and then $v_1 = (1, 2, 0)$ and $v_2 = (-1, 2, 3)$ will be the columns of the given matrix for T. Since v_1 and v_2 are linearly independent, we can complete them to a basis. To do this we just need to find a third vector of $\mathbb{R}^3$ that is not a linear combination of v_1 and v_2. For instance, $v_3 = e_3 = (0, 0, 1)$ works.

4. Let A be an $m \times n$ matrix with entries in F. The different row operations that can be performed on A are

   - $R1(i, a)$: Multiply the i^{th} row of A by a nonzero scalar $a \in F$. $(1 \leq i \leq m)$
   - $R2(i, j)$: Swap the i^{th} and j^{th} rows of A. $(1 \leq i, j \leq m)$
   - $R3(i, j)$: Add the i^{th} row of A to the j^{th} row of A. $(1 \leq i, j \leq m)$

   For each row operation R listed above, exhibit an $m \times m$ matrix X such that XA is the matrix obtained by applying the row operation R to A. Verify that each such X is invertible. (Hint: think about which row operation would undo R.)

   Solution. Given a row operation R, assume that there is a matrix X such that XA is the matrix obtained by applying the row operation R to A for any A. Then $X = XI$ is the matrix we get by performing the row operation R to the identity matrix I. Hence the matrices we get, and their inverses, are:

   - $R1(i, a)$: X is the matrix with 1's down the diagonal, except in the i^{th} row where the entry is a. In other words, $X_{jk} = 0$ if $j \neq k$; $X_{jj} = 1$ if $j \neq i$; and $X_i i = a$. X^{-1} corresponds to the inverse row operation, which will be $R1(i, 1/a)$ – dividing the i^{th} row by a. Hence X^{-1} will have $1/a$ in the i^{th} row instead of a.
   - $R2(i, j)$: X is the matrix obtained from I by swapping rows i and j. Hence

$$X_{kl} = \begin{cases} 1, & k = l \neq i, j \\ 1, & \{k, l\} = \{i, j\} \\ 0, & \text{otherwise} \end{cases}$$

The inverse of X corresponds to the same row operation – swapping rows i and j back – and hence $X^{-1} = X$.

- $R3(i, j)$: X is obtained by adding the i^{th} row of I to the j^{th} row. Thus

$$X_{kl} = \begin{cases} 1, & k = l \\ 1, & k = j, l = i \\ 0, & \text{otherwise} \end{cases}$$

  The inverse of X corresponds to subtracting the i^{th} row from the j^{th} row, and thus the matrix X^{-1} will look the same as X but with -1 in the (j, i)-entry, instead of $+1$.