Math 108A - Home Work # 5 Solutions

LADR Problems, p. 59-60:

4. We must show that $V = null(T) + Fu$ and also that $null(T) \cap Fu = \{0\}$. First suppose, $v \in null(T) \cap Fu$. This means that $v = au$ for some $a \in F$ and $Tv = 0$. Thus $T(au) = T(v) = 0$, which implies that $aT(u) = 0$. Since $T(u) \neq 0$ by assumption, we must have $a = 0$. Thus $v = 0u = 0$, and we conclude that $null(T) \cap Fu = \{0\}$.

   Now let $v \in V$. To produce a vector in $null(T)$, consider $T(v)$ and $T(u)$, which are two vectors in the one-dimensional vector space F. Hence $T(v)$ and $T(u)$ must be linearly dependent, which means that $T(v) = aT(u) = T(au)$ for some $a \in F$. Hence $T(v - au) = 0$, so $v - au \in null(T)$. We now have $v = (v - au) + au \in null(T) + Fu$. This shows that $V = null(T) + Fu$.

5. Assume that $\{v_1, \ldots, v_n\}$ is a linearly independent set of vectors in V and $T : V \to W$ is an injective linear map. If $c_1T(v_1) + \cdots c_nT(v_n) = 0$ for some $c_i \in F$, then by linearity of T, we have $T(c_1v_1 + \cdots + c_nv_n) = 0$. Since we assumed that T is injective, $c_1v_1 + \cdots + c_nv_n \in null(T) = \{0\}$, which means that $c_1v_1 + \cdots + c_nv_n = 0$. By linear independence of $\{v_1, \ldots, v_n\}$ we conclude that $c_i = 0$ for all i. This shows that $\{Tv_1, \ldots, Tv_n\}$ is linearly independent.

7. Assume that $span(v_1, \ldots, v_n) = V$ and $T : V \to W$ is a surjective linear map. If $w \in W$, there exists a $v \in V$ such that $Tv = w$. We can write $v = a_1v_1 + \cdots + a_nv_n$ since the vectors v_i span V. Now, by linearity of T, we have $w = T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n)$, which shows that w is in the span of $\{T(v_1), \ldots, T(v_n)\}$. Since w was arbitrary, we see that $span(Tv_1, \ldots, Tv_n) = W$.

9. Since $T : F^4 \to F^2$ is a linear map, by the Rank-Nullity Theorem, we know that

$$4 = \dim F^4 = \dim\ null(T) + \dim\ range(T).$$

   We claim that $null(T)$ is 2-dimensional. One easily checks that it is spanned by $(5, 1, 0, 0)$ and $(0, 0, 7, 1)$, and these two vectors are clearly linearly independent since neither is a scalar multiple of the other. Thus $\{(5, 1, 0, 0), (0, 0, 7, 1)\}$ is a basis for $null(T)$ and $\dim\ null(T) = 2$. (In fact, it is only necessary to check that these 2 basis vectors span $null(T)$, so that we know $\dim\ null(T) \leq 2$.) The above equality now implies that $\dim\ range(T) = 2$, and since $range(T)$ is a subspace of F^2, which also has dimension 2, we know that $range(T) = F^2$. Thus T is surjective.

10. Suppose that $T : F^5 \to F^2$ is a linear map with null space

$$\ker(T) = U = \{\vec{x} \in F^5 \mid x_1 = 3x_2, x_3 = x_4 = x_5\}.$$

The rank-nullity theorem tells us that $\dim U = \dim F^5 - \dim \operatorname{Im}(T) \geq 3$ since the image of T is a subspace of F^2 and hence has dimension at most 2, However,

$$U = \{(3x_2, x_2, x_3, x_3, x_3) \mid x_2, x_3 \in F\} = \mathit{span}\{(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)\}.$$

Since the two vectors in this spanning set are clearly linearly independent, $\dim U = 2$, a contradiction.