Name: ________________

Perm No.: _____________

Math 108A - Final Exam

June 8, 2009

Instructions:

- This exam consists of 5 problems for a total of 60 points, plus one extra credit problem worth up to 10 points. The exact point-value of each part is stated next to its number.
- You must show all your work and fully justify your answers in order to receive full credit. You may use any theorem/fact from lecture or from the text (Ch. 1–5). But your work should clearly indicate which result you are using and, if necessary, explain how you are using it.
- Partial credit will be given for work that is relevant and correct.
- You may assume the results of earlier questions are true, even if you can't prove them, in order to do later questions (eg. to do part (c), you may assume parts (a) and (b)).
- No books, notes or calculators are allowed.
- Write your answers on the test itself, in the space alotted. Scratch paper is available if you need it. You may want to work out your solutions first on scratch paper, so that you can write them on the test as neatly as possible. You may attach additional pages if necessary.

1	
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Total	
6	

1. (10 points) Let $U = \{(x,y,z,w) \in F^4 \mid x+y+z = w\}$.

(a) (6 pts) Find the dimension of U. Justify your answer.

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \\ x+y+z \end{pmatrix} \in F^4 \;\middle|\; x,y,z \in F \right\} = \left\{ x\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \;\middle|\; x,y,z \in F \right\}$$

$$= \operatorname{Span}\{\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}}\}$$

L.I. $\Rightarrow$ these are a basis for U

$\Rightarrow \dim U = 3$

if $c_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

then $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_1+c_2+c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ all $c_i = 0$!

(b) (4 pts) Show that for *every* linear map $T : F^4 \to F^2$, there is a nonzero vector $u \in U$ such that $T(u) = 0$.

Let $T : F^4 \to F^2$.

Rank-Nullity Thm $\Rightarrow$ $\dim \ker(T) = \dim F^4 - \dim \operatorname{Im}(T)$

$= 4 - \dim \operatorname{Im}(T)$

$\geq 4 - 2 = 2.$

where we have used $\dim \operatorname{Im}(T) \leq 2$, since $\operatorname{Im}(T)$ is a subspace of F^2.

Another theorem $\Rightarrow$ $\dim(U + \ker(T)) = \dim U + \dim \ker(T) - \dim(U \cap \ker(T))$.

$U + \ker(T) \subseteq F^4 \Rightarrow \dim(U + \ker(T)) \leq 4.$

$\dim U = 3, \quad \dim \ker(T) \geq 2$

$\Rightarrow \dim(U \cap \ker(T)) = -\dim(U + \ker(T)) + \dim U + \dim \ker(T)$

$\geq -4 + 3 + 2 = 1.$

Thus $U \cap \ker(T) \neq \{\vec{0}\}$ & any nonzero $\vec{u} \in U \cap \ker(T)$ satisfies $T(\vec{u}) = \vec{0}$.

2. (10 points) Let $T : V \to V$ be a linear map.

(a) (4 pts) Complete the definitions by filling in the blanks.

A scalar $\lambda \in F$ is an **Eigenvalue** of T if $\ker(T - \lambda I)$ $\neq \{\vec{0}\}$.

A vector $v \in V$ is an **Eigenvector** of T is there exists $\lambda \in F$

such that $T\vec{v} = \lambda\vec{v}$ and $\vec{v} \neq \vec{0}$.

(b) (6 pts) Suppose that the eigenvalues of T are 1 and 2. What are the eigenvalues of $T - I_V$? Explain your answer. (Here, $I_V : V \to V$ is the identity map.)

If 1 & 2 are the eigenvalues of T,

$T\vec{v} = \lambda\vec{v}$ $(\vec{v} \neq \vec{0})$ $\Rightarrow$ $\lambda = 1$ or 2.

Suppose λ = eigenvalue of $T - I$.

$$(T - I)(\vec{v}) = \lambda\vec{v} \Rightarrow T(\vec{v}) - \vec{v} = \lambda\vec{v}$$

$$\Rightarrow T(\vec{v}) = (\lambda + 1)\vec{v}$$

$$\Rightarrow \lambda + 1 = \text{eigenvalue of } T = 1 \text{ or } 2$$

$$\Rightarrow \boxed{\lambda = 0 \text{ or } 1}$$

3. (10 points) Let $\mathcal{E}$ be the standard basis for F^2, and let $\mathcal{B}$ be the basis $\{(1,4),\ (2,9)\}$. Let $T: F^2 \to F^2$ be the linear map defined by

$$T(x,y) = (2x-y, x-2y), \quad \forall\ x,y \in F.$$

(a) (4 pts) Find the coordintates of each of the standard basis vectors in the basis $\mathcal{B}$.

$\vec{e}_1 = (1,0) = c_1(1,4) + c_2(2,9) = (c_1+2c_2, 4c_1+9c_2)$

$c_1 + 2c_2 = 1 \qquad \Rightarrow c_1 = 9$

$4c_1 + 9c_2 = 0 \Rightarrow c_2 = -4/9\, c_1 \qquad \Rightarrow c_2 = -4$

$$\boxed{[\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}}$$

$\vec{e}_2 = (0,1) = c_1(1,4) + c_2(2,9) = (c_1+2c_2, 4c_1+9c_2)$

$c_1 + 2c_2 = 0 \Rightarrow c_1 = -2c_2 \qquad \Rightarrow c_1 = -2.$

$4c_1 + 9c_2 = 1 \qquad \Rightarrow c_2 = 1$

$$\boxed{[\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

(b) (6 pts) Find the matrix $Mat(T;\mathcal{B})$ for T relative to the basis $\mathcal{B}$.

$$Mat(T;\mathcal{B}) = C^{-1}AC = \underbrace{\begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix}}_{Mat(I;\mathcal{E},\mathcal{B})} \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & -2 \end{pmatrix}}_{Mat(T;\mathcal{E})} \underbrace{\begin{pmatrix} 1 & 2 \\ 4 & 9 \end{pmatrix}}_{Mat(I;\mathcal{B},\mathcal{E})}$$

$$= \begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} -2 & -5 \\ -7 & -16 \end{pmatrix} = \boxed{\begin{pmatrix} -4 & -13 \\ 1 & 4 \end{pmatrix}}$$

4. (20 points) Let $T : F^3 \to F^3$ be defined by $T(x, y, z) = (y, x + y + z, y)$.

(a) (3 pts) Find the matrix A of T with respect to the standard basis $\mathcal{E}$ of F^3.

$$A = \mathrm{Mat}(T; \mathcal{E}) = \left(\begin{array}{ccc} | & | & | \\ T(1,0,0) & T(0,1,0) & T(0,0,1) \\ | & | & | \end{array} \right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) (6 pts) Find a basis for $\ker(T)$, and a basis for $\mathrm{im}(T)$.

$$\ker(T) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \;\middle|\; \begin{array}{l} y = 0 \\ x + y + z = 0 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \;\middle|\; \begin{array}{l} y = 0 \\ z = -x \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} \;\middle|\; x \in F \right\} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

Hence $\boxed{\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}}$ is a basis for $\ker(T)$.

$$\mathrm{Im}(T) = \text{Column Space of } A = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\underbrace{\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}}_{\text{clearly L.I.}} \Rightarrow \boxed{\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}} \text{ is a Basis for } \mathrm{Im}(T).$$

(c) (3 pts) What are the eigenvalues of T?

$$p(\lambda) = \det\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda\big(-\lambda(1-\lambda)-1\big) - 1(-\lambda)$$

$$= -\lambda(\lambda^2 - \lambda - 1) + \lambda$$

$$= -\lambda^3 + \lambda^2 + 2\lambda = -(\lambda-2)(\lambda+1)\lambda$$

$$\Rightarrow \boxed{\lambda = 0, -1, 2}$$

(d) (3 pts) Find an eigenvector for each eigenvalue of T.

$\lambda = 0$: An eigenvector is simply a null vector $0 \neq \vec{v} \in \ker(T)$. eg. $\boxed{\vec{v}_0 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$ from (b).

$\lambda = -1$: $T - \lambda I = T + I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$$\rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \ker(T+I) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in F^3 \,\middle|\, x - z = 0,\ y + z = 0 \right\}.$$

$\Rightarrow \boxed{\vec{v}_{-1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}$ is an eigenvector

$\lambda = 2$ $T - \lambda I = T - 2I$

$$= \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & -1 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x = z$, $y = 2z$ $\boxed{\vec{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$.

(e) (5 pts) Compute the matrix A^{20}. (A is just the matrix of T from part (a)).

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

A has 3 distinct Eigenvalues $\Rightarrow$ A is diagonalizable.

$\mathcal{B} = \{\vec{v}_0, \vec{v}_{-1}, \vec{v}_2\}$ is a basis of eigenvectors.

$$\& \ \operatorname{Mat}(T;\mathcal{B}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = C^{-1}AC, \text{ where } C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

Thus $A = C\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}C^{-1}$ $\&$ $A^{20} = C\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}^{20} C^{-1}$

$$= C\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{20} \end{pmatrix}C^{-1}.$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{20} \end{pmatrix} C^{-1}$$

$$= \begin{pmatrix} 0 & 1 & 2^{20} \\ 0 & -1 & 2^{21} \\ 0 & 1 & 2^{20} \end{pmatrix}\begin{pmatrix} 3 & 0 & -3 \\ 2 & -2 & 2 \\ 1 & 2 & 1 \end{pmatrix} / 6$$

$$= \boxed{\begin{pmatrix} 2+2^{20} & -2+2^{21} & 2+2^{20} \\ -2+2^{21} & -2+2^{22} & -2+2^{21} \\ 2+2^{20} & -2+2^{21} & 2+2^{20} \end{pmatrix} / 6.}$$

Let's find C^{-1} by Row reduction:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 2 & 1 \\ 0 & 1 & 4 & 1 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 1/6 & 1/3 & 1/6 \end{array}\right) = C^{-1}.$$

5. (10 points) Let $\mathcal{P}_3(F)$ be the vector space of polynomials with coefficients in F and degree ≤ 3; and let $T : \mathcal{P}_3(F) \to \mathcal{P}_3(F)$ be defined by

$$T(f(x)) = f(x+1), \quad \text{for all } f(x) \in \mathcal{P}_3(F).$$

For example $T(x^2 + x) = (x+1)^2 + (x+1)$.

(a) (3 pts) Show that T is a linear map.

- $T(cf(x)) = cf(x+1) = cT(f(x)) \quad \forall c \in F \quad \forall f \in \mathcal{P}_3(F).$
- $T(f(x)+g(x)) = f(x+1) + g(x+1) = T(f(x)) + T(g(x))$
  $\forall f, g \in \mathcal{P}_3(F).$

$\Rightarrow T$ - linear.

(b) (3 pts) Compute $Mat(T; \mathcal{B})$ where $\mathcal{B}$ is the basis $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3(F)$.

$$Mat(T;\mathcal{B}) = \begin{pmatrix} | & | & | & | \\ T(1)_{\mathcal{B}} & T(x)_{\mathcal{B}} & T(x^2)_{\mathcal{B}} & T(x^3)_{\mathcal{B}} \\ | & | & | & | \end{pmatrix}$$

$$\left.\begin{aligned} T(1) &= 1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \\ T(x) &= x+1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \\ T(x^2) &= (x+1)^2 = x^2 + 2x + 1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}} \\ T(x^3) &= (x+1)^3 = x^3 + 3x^2 + 3x + 1 \\ &= \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}_{\mathcal{B}} \end{aligned}\right\} \Rightarrow \boxed{Mat(T;\mathcal{B}) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

(c) (4 pts) Find all eigenvalues and eigenvectors of T.

$\text{Mat}(T;\mathcal{B})$ = Upper triangular

$\Rightarrow$ the eigenvalues of T are the diagonal entries of this matrix

$\Rightarrow$ $\boxed{\lambda = 1}$ is the only eigenvalue of T.

To find eigenvectors, we consider the matrix for $T - \lambda I = T - I$: $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and compute its kernel:

$$\ker(T-I) = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{\mathcal{B}} \;\middle|\; \underbrace{\begin{matrix} b+c+d=0 \\ 2c+3d=0 \\ 3d=0 \end{matrix}}_{\Rightarrow\, b=c=d=0} \right\}$$

$$= \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}} \;\middle|\; a \in F \right\} = \text{Span}\{1\}$$

$\Rightarrow$ $f(x) = a$, $a \neq 0$ (ie. constant polynomials)

are the only eigenvectors of T.

6. (Extra Credit: 10 points) Let $V = \mathcal{C}(\mathbb{R})$ be the vector space of continuous functions $f : \mathbb{R} \to \mathbb{R}$, and define the linear map $T : V \to V$ by $T(f(x)) = f(x+1)$ as in the previous question.

Find all eigenvalues of T and justify your answer. (Hint: Exponential and Trig functions.)

- Suppose $\lambda > 0$:

  Let $f(x) = \lambda^x$.

  $T(f(x)) = f(x+1) = \lambda^{x+1} = \lambda \cdot \lambda^x = \lambda f(x)$.

  $\Rightarrow \lambda =$ eigenvalue of T.

- Suppose $\lambda = 0$:

  If $T(f(x)) = f(x+1) = 0 \cdot f(x) = 0$

  then $f(x+1) = 0 \quad \forall x \in \mathbb{R}$.

  $\Rightarrow f(x) = 0. \quad \forall x \in \mathbb{R}$.

  $\Rightarrow \lambda = 0$ is <u>Not</u> an Eigenvalue of T.

- Suppose $\lambda < 0$.

  Let $\lambda = -a, \quad a > 0$

  We want $f(x+1) = -a f(x) \quad \forall x \in \mathbb{R}$.

  take $f(x) = a^x \sin(\pi x)$

  $$f(x+1) = a^{x+1} \sin(\pi x + \pi) = a \cdot a^x (-\sin(\pi x))$$
  $$= -a f(x).$$

  $\Rightarrow \lambda = -a$ is an Eigenvalue of T.

$\therefore$ Eigenvalues of T are all elements of $\boxed{\mathbb{R} - \{0\}.}$