MATH 220B: SOME PROBLEMS
APRIL 6, 2018

You may find it helpful to attempt the problems below.

(1) Let R be an integral domain. Let r be an element of R which is non-zero, is not a unit, and cannot be written as a finite product of irreducible elements. Show that r can be written as a product $r = st$ where the principal ideal (r) is strictly contained in the principal ideal (s) and s is non-zero, is not a unit, and cannot be written as a finite product of irreducible elements. Hence show that in a Noetherian integral domain, every element which is neither zero nor a unit can be written as a finite product of irreducible elements. [**Hint:** You may find it helpful to read Chapter 8 of Dummit and Foote first.]

(2) Dummit and Foote, Section 10.4, nos. 4, 5, 6, 11, 14, 15, 24, 25.

(3) Dummit and Foote, Section 10.5, nos. 21, 22, 25, 26.

(4) Dummit and Foote, Section 15.4, no. 16.

(5) (i) Let E and F be fields. Show that E and F have subfields E_1 and F_1 respectively such that $E_1 \simeq F_1$ if and only if E and F have the same characteristic.

(ii) Suppose that E and F are extension fields of the same field k. Show that the tensor product $E \otimes_k F$ is a non-zero k-vector space, and that it becomes a ring when multiplication is defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$; verify that it contains isomorphic copies of E and F.

(iii) Hence show that there exists a field Ω having subfields isomorphic to E and F respectively.

(iv) Deduce that two fields can be embedded in a common extension field if and only if they have the same characteristic.

[**Hint:** An ideal I in a ring R is maximal if and only if R/I is a field. Every proper ideal in R is contained in a maximal ideal.]

(6) Let R be the ring of numbers of the form $m + \sqrt{-6}$, where m and n are rational integers, and let M be the subset consisting of numbers of the form $2m + n\sqrt{-6}$. Show that M is an R-module, and verify that M is not a free R-module.

By calculating generators for $M \otimes_R M$ or otherwise, show that $M \otimes_R M$ is free.

(7) Let V be the space of polynomials

$$p(x) = a_n X^n + a_{n-1} X^{n-1} + ... + a_1 X + a_0$$

with complex coefficients and of degree $\leq n$; the endomorphism $\Delta : V \to V$ is defined by

$$(\Delta p)(X) = p(X + 1) - p(X).$$

What is the Jordan normal form for Δ?