MATH 220A: PROBLEM SHEET II
OCTOBER 11, 2017

Answers should be **neatly** typed on 8.5-by-11 inch paper. Please use only one side of each sheet of paper.

(1) Let G be a finite group containing a subgroup H of index p, where p is the smallest prime divisor of $|G|$. Prove that H is a normal subgroup of G.

(2) (i) If a group G acts on sets S and T, we can define a group action on the Cartesian product $S \times T$ by the rule $g(x, y) = (gx, gy)$. Show that the stabiliser of (x, y) is $A \cap Y$, where A is the stabiliser of x and B is the stabiliser of y.
(ii) Let G be a group and H, K subgroups of index r, s respectively. Show that $H \cap K$ has index at most rs in G.

(3) Show that any group G of order $4n + 2$ has a subgroup of index 2. (Hint: Show that G has an element of order 2. Now apply Cayley's theorem.)

(4) (i) Show that a finite simple group whose order is at least $r!$ cannot have a proper subgroup of index r.
(ii) Let $n \geq 5$, and assume that A_n is simple. Show that S_n has no proper subgroup of index less that n, other than A_n.

(5) Show that every finite group can be represented as a group of even permutations.

(6) Let G be a group, and let $H \leq G$. The group H is said to be *intravariant* in G if, for every automorphism α of G, α maps H to a conjugate of H in G.
(i) If G is finite, show that every Sylow subgroup is intravariant in G.
(ii) Let K be a normal subgroup of G. Suppose that H is a subgroup of K which is intravariant in K. Prove that $G = N_G(H)K$. (This generalises the Frattini argument.)

(7) Let G be a finite non-abelian simple group, and let p be a prime divisor of $|G|$. Show that the number n of Sylow p-subgroups of G is greater than 1.

(8) Let G be a finite group, p a prime divisor of $|G|$, and n the number of distinct Sylow p-subgroups of G. Prove that the normalisers in G of the Sylow p-subgroups of G form a single conjugacy class of n subgroups of G.