MATH 115B SOLUTION SET VI
JUNE 11, 2007

(1) Show that there are infinitely many even integers n with the property that both $n+1$ and $(n/2)+1$ are perfect squares. Exhibit two such integers.

Solution:

For an integer n, suppose that $n+1 = u^2$ and $(n/2)+1 = v^2$ for some u and v. Then

$$u^2 - 2v^2 = (n+1) - (n+2) = -1.$$

Now $\sqrt{2} = [1; \overline{2}]$, and the convergents $C_{2k} = p_{2k}/q_{2k}$ orf $\sqrt{2}$ provide solutions $u = p_{2k}$, $v = q_{2k}$ of the equation $u^2 - 2v^2 = -1$. Thus there are infinitely many integers n for which $n+1$ and $(n/2)+1$ are boths quares. Two such n are 48 and 1680.

(2) Find the fundamental solutions of the following equations:
(i) $x^2 - 29y^2 = 1$;
(ii) $x^2 - 41y^2 = 1$.

Solution:

(i) Observe that $\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$ has period 5. Hence a fundamental solution of the given equation is obtained from the convergent C_9 of $\sqrt{29}$, namely $C_9 = 9801/1820$ gives $x = 9801$, $y = 1820$.

(ii) We have that $\sqrt{41} = [6; \overline{2, 2, 12}]$ has period 3. The convergent $C_5 = 2049/320$ gives the fundamental solution $x = 2049$, $y = 320$ of the given equation.

(3)(a) Prove that whenever the equation $x^2 - dy^2 = c$ is soluble, then it has infinitely many solutions.
[Hint: If u, v satisfy $x^2 - dy^2 = c$ and r, s satisfy $x^2 - dy^2 = 1$, then

$$(ur \pm dvs)^2 - d(us \pm vr)^2 = (u^2 - dv^2)(r^2 - ds^2) = c.]$$

(b) Given that $x = 16$, $y = 6$ is a solution of $x^2 - 7y^2 = 4$, find two other positive solutions.

Solution:

(a) Assume that the equation $x^2 - dy^2 = c$ has a solution u, v, and that r, s satisfies $r^2 - ds^2 = 1$. This implies that

$$\begin{aligned} &(ur + dvs)^2 - d(us + vr)^2 \\ &= u^2(r^2 - ds^2) - dv^2(r^2 - ds^2) \\ &= (u^2 - dv^2)(r^2 - ds^2) = ccdot1 = c, \end{aligned}$$

whence $x = ur + dvs$, $y = us + vr$ is another solution of $x^2 - dy^2 = c$. Since there are infinitely many solutions to $x^2 - dy^2 = 1$, it follows that there are infinitely many solutions to $x^2 - dy^2 = c$.

(b) The equation $x^2 - 7y^2 = 4$ is satisfied by $x = 16$, $y = 6$. Since $x = 8$, $y = 3$ and $x = 127$, $y = 48$ are both solutions of $x^2 - 7y^2 = 1$, part (a) shows that two other solutions of $x^2 - 7y^2 = 4$ are

$$x = 16 \cdot 8 + 7 \cdot 6 \cdot 3 = 254$$
$$y = 16 \cdot 3 + 6 \cdot 8 = 96,$$

and

$$x = 18 \cdot 71 + 35 \cdot 3 \cdot 12 = 2538$$
$$y = 18 \cdot 12 + 3 \cdot 71.$$