Math 3A Fall Quarter Final Examination
December 11, 2007

NAME: Answer Key & Marking Scheme

TA & DISCUSSION SECTION:

You have three hours in which to complete this examination. Attempt all of the questions. **Note that you will not be awarded full credit on a question unless your answer is clearly, carefully and neatly stated and explained.**

Problem	Maximum Score	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
Total	35	

(1)(i) Evaluate $\lim_{\theta\to 0^+} \dfrac{\sin\theta}{2\sin 2\theta - \sin\theta}$

1 pt.

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{2\sin 2\theta - \sin\theta} \overset{(H)}{=} \lim_{\theta\to 0^+} \frac{\cos\theta}{4\cos 2\theta - \cos\theta}$$

$$= \frac{\lim_{\theta\to 0^+} \cos\theta}{\lim_{\theta\to 0^+} (4\cos 2\theta - \cos\theta)}$$

$$= \frac{1}{3}$$

(ii) Evaluate $\lim_{x\to\infty} (xe^{1/x} - x)$.

2 pts.

$$\lim_{x\to\infty} (xe^{1/x} - x) = \lim_{x\to\infty} \frac{e^{1/x} - 1}{1/x}$$ 1 pt

$$\overset{(H)}{=} \lim_{x\to\infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2}$$

$$= \lim_{x\to\infty} e^{1/x}$$

$$= 1$$ 1 pt.

(iii) If $f'(x)$ is continuous, $f(2) = 0$ and $f'(2) = 7$, evaluate

$$\lim_{x\to 0} \frac{f(2+3x) + f(2+5x)}{x}$$

2 pts.

[Since $f(2) = 0$, the given limit is of the form $0/0$]

$$\lim_{x\to 0} \frac{f(2+3x) + f(2+5x)}{x} \overset{(H)}{=} \lim_{x\to 0} \frac{f'(2+3x)\cdot 3 + f'(2+5x)\cdot 5}{1}$$ 1 pt.

$$= f'(2)\cdot 3 + f'(2)\cdot 5$$

$$= 21 + 35$$

$$= 56$$ 1 pt.

(2) Find the derivatives with respect to x of the following functions. Simplify your answers as much as possible.

(i) $y = e^{x^2}$

1 pt. $$\frac{dy}{dx} = 2x\, e^{x^2}$$

(ii) $y = x\sin(1/x)$

2 pts.

$$\frac{dy}{dx} = 1.\, \mathrm{Sin}(1/x) + x \,\mathrm{Cos}\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}$$ 1pt

$$= \mathrm{Sin}\left(\frac{1}{x}\right) - \frac{1}{x}\mathrm{Cos}\left(\frac{1}{x}\right)$$ 1pt

(iii) $y = x^x$

$$y = x^x$$
$$\Rightarrow \log y = x \log x$$
$$\Rightarrow y' \cdot \frac{1}{y} = \log x + x \cdot \frac{1}{x}$$ 1 pt.

$$\Rightarrow y' = y(\log x + 1)$$
i.e. $$y' = x^x(\log x + 1)$$ 1 pt.

3 pts. (3)(a) Find the equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

$g(5) = -3 \Rightarrow$ the point $(5, -3)$ lies on the graph.
$g'(5) = 4 \Rightarrow$ the slope of the tangent line at $x = 5$ is 4. 1 pt.

$\therefore$ Equation of tangent line is 1 pt.

$$\frac{y - (-3)}{x - 5} = 4$$

$$\Rightarrow \quad y = 4x - 23.$$ 1 pt.

2 pts. (b) If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

Since $(4, 3)$ is on the graph of $y = f(x)$, we have that $f(4) = 3$. 1 pt

$f'(4)$ = slope of tangent line to the graph when $x = 4$
= slope of line ~~joing~~ joining $(4, 3)$ and $(0, 2)$ 1 pt.

$$= \frac{3 - 2}{4 - 0}$$

$$= \frac{1}{4}$$

2 pts. (4)(a) Find y' if $x^2 - xy + y^2 = 3$.

$$\left.\begin{aligned} & x^2 - xy + y^2 = 3 \\ \Rightarrow\ & 2x - y - xy' + 2yy' = 0 \end{aligned}\right] \text{ 1 pt.}$$

$$\Rightarrow \quad \left. y' = \frac{y - 2x}{2y - x}. \right] \text{ 1 pt.}$$

3 pts. (b) Find all the points at which the tangent line to $x^2 - xy + y^2 = 3$ is horizontal.

$y' = 0$ when $y - 2x = 0$.] 1 pt.

Substituting $y = 2x$ into the equation $x^2 - xy + y^2 = 3$ gives

$$x^2 - x(2x) + (2x)^2 = 3$$

$$\Rightarrow \quad 3x^2 = 3$$

$$\Rightarrow \quad x = \pm 1.$$

] 1 pt.

$\therefore$ The tangent line is horizontal at the points $(1, 2)$ and $(-1, -2)$.] 1 pt.

(5) Suppose that the derivative of a function $f(x)$ is $f'(x) = (x+1)^2(x-3)^5(x-6)^4$ for all values of x.

2 pts. (a) Find all the intervals on which $f(x)$ is increasing. (You must justify your answer.)

$f(x)$ is increasing when $f'(x) > 0$. 1 pt.

We have that $f'(x) > 0$ only when $x > 3$.
$\therefore f(x)$ is increasing only on the interval $(3, \infty)$ 1 pt.

1 pt. (b) Which values of x are critical numbers of $f(x)$?

The critical numbers of $f(x)$ are
$x = -1, 3, 6$. 1 pt.

2 pts. (c) Find all values of x for which $f(x)$ has a local minimum. (You must justify your answer.)

$f'(x)$ changes sign from negative to positive only at $x = 3$. 1 pt.

$\therefore f(x)$ has a local minimum only at $x = 3$. 1 pt.

(6)(a) Write down a formula for the surface area S of a sphere of radius r.

$$S = 4\pi r^2$$ 2 pts.

(b) A spherical snowball melts in such a way that its surface area decreases at a rate of $1\,cm^2/min$. Find the rate at which the diameter of the snowball is decreasing when the diameter is $10\,cm$.

Let x denote the diameter of the snowball. Then $r = \frac{1}{2}x$, and so 1 pt.

$$S = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2$$

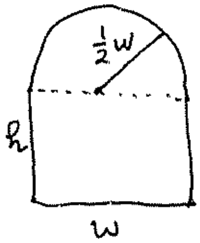
$$\therefore \frac{dS}{dt} = 2\pi x \frac{dx}{dt}$$ 1 pt.

$$\therefore -1 = 2\pi(10) \cdot \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{-1}{20\pi}$$ 1 pt.

Therefore the diameter of the snowball is decreasing at a rate of $\frac{1}{20\pi}$ cm/min.

(7) A Norman window has the shape of a rectangle (of width w and height h) surmounted by a semicircle. (Thus the diameter of the semicircle is equal to the width w of the rectangle.) If the perimeter of the window is 6 metres, find the dimensions of the rectangle (i.e. the values of w and h) so the the greatest possible amount of light is admitted through the window.



As the perimeter of the window is 6m, we have

$$2h + w + \pi\left(\frac{w}{2}\right) = 6$$

$$\Rightarrow h = \frac{1}{2}\left(6 - w - \frac{\pi w}{2}\right) = 3 - \frac{w}{2} - \frac{\pi w}{4}$$

1 pt.

A = area of window

= (area of rectangle) + (area of semicircle)

$$= hw + \frac{1}{2}\pi\left(\frac{w}{2}\right)^2$$

$$= \left(3 - \frac{w}{2} - \frac{\pi w}{4}\right)w + \frac{\pi w^2}{8}$$

$$= 3w - \frac{w^2}{2} - \frac{\pi w^2}{8}$$

1 pt

$$\therefore \frac{dA}{dw} = 3 - w - \frac{\pi w}{4} = 3 - \left(1 + \frac{\pi}{4}\right)w.$$

1 pt

$$\therefore \frac{dA}{dw} = 0 \Rightarrow w = \frac{3}{1 + \pi/4} = \frac{12}{4 + \pi}$$

$$\therefore h = 3 - \frac{6}{4+\pi} - \frac{3\pi}{4+\pi} = \frac{6}{4+\pi}$$

1 pt

Also $\dfrac{d^2A}{dw^2} = -\left(1 + \dfrac{\pi}{4}\right) < 0$, and so these values of h and w give a ~~local~~ global maximum.

1 pt.