CCS INTRODUCTION TO ANALYSIS: PROBLEM SHEET IV
FEBRUARY 6, 2006

Answers should be **neatly** written out on letter-sized paper. Please write on only one side of each sheet of paper, and please begin each new answer on a fresh page.

(1) Let (a_n) $(n = 1, 2, ...)$ be a decreasing sequence of positive real numbers. By using the General Principle of Convergence, or otherwise, prove that if there exists a positive number k such that $a_n \geq k/n$ for infinitely many n, then $\sum_{n=1}^{\infty} a_n$ diverges.

Give an example of a decreasing sequence (a_n) of positive numbers such that $na_n \to 0$ as $n \to \infty$, and $\sum_{n=1}^{\infty} a_n$ diverges. Show that your example has these properties.

(2) Give either a proof or a counterexample for each of the following:

(i) If the sequence (a_n) $(n = 1, 2, ...)$ of positive numbers tends to 0 as $n \to \infty$, then $\sum_{n=0}^{\infty}(-1)^n a_n^{1/n}$ converges.

(ii) If the sequence (a_n) of real numbers is such that, for every positive integer p, $|a_{n+p} - a_n| \to 0$ as $n \to \infty$, then (a_n) converges.

(3) Let (a_n) be a decreasing sequence of positive real numbers such that $a_n \to 0$ as $n \to \infty$. Prove that $\sum_{n=1}^{\infty} b_n$ is convergent, where

$$b_n = \frac{a_1 + a_2 + ... + a_n}{n}$$

for $n = 1, 2, ...$

(4) Let (x_n) be a sequence of real numbers, with $\lim_{n\to\infty} x_n = 0$, and let $s_n = \sum_{k=1}^{n} x_k$. Prove that

(a) $n^{-1}s_n \to 0$ as $n \to \infty$;

(b) if (x_n) is monotonically decreasing, and if (nx_n) does not converge to zero, then $s_n \to \infty$ as $n \to \infty$.

Deduce that $\sum 1/n$ diverges.

(5) Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(1 + x)(1 + 2x)...(1 + nx)}{(1 + x)(2 + x)...(n + x)}$$

for all positive values of x. You should state precisely any tests for convergence that you use.

(6) The sequences (a_n), (b_n) of real numbers are such that $a_n \neq 0$ or 1, $\sum_{n=1}^{\infty} a_n$ is convergent, and $b_n/a_n \to 1$ as $n \to \infty$. If $a_n > 0$ for all n, prove that the series $\sum_{n=1}^{\infty} a_n^2$, $\sum_{n=1}^{\infty} a_n/(1 - a_n)$, and $\sum_{n=1}^{\infty} b_n$ are convergent. Which, if any, of these series necessarily converges if the restriction $a_n > 0$ is lifted? Give proofs or counterexamples as appropriate.