MATH 225A PROBLEM SHEET III
FEBRUARY 6, 2006

1. Let p be an odd prime, and let ζ_p be a primitive pth root of unity. Set $\Gamma = \mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$, and let $\chi : \Gamma \to \mathbf{C}^*$ be a character of order $n > 1$ (i.e. χ is a group homomorphism, and n is the least integer such that χ^n is the trivial homomorphism). We define the Gauss sum $\tau(\chi, \zeta_p)$ by

$$\tau(\chi, \zeta_p) = \sum_{\gamma \in \Gamma} \chi(\gamma)\zeta_p^{\gamma}.$$

(i) Show that, for $\gamma \in \Gamma$, we have

$$\tau(\chi, \zeta_p^{\gamma}) = \chi(\gamma^{-1})\tau(\chi, \zeta_p).$$

(ii) Show that

$$\tau(\chi, \zeta_p)\overline{\tau(\chi, \zeta_p)} = p.$$

(Here $\overline{z}$ denotes the complex conjugate of z.)

(iii) Let χ be the unique character of Γ of order 2. From (i) and (ii), deduce that

$$\tau(\chi, \zeta_p) = \pm\sqrt{\left(\frac{-1}{p}\right)p}.$$

2. Describe the factorisation of the ideals generated by $2, 3, 5$ in $\mathbf{Q}(\sqrt[3]{6})$.

3. Let θ satisfy $\theta^3 - \theta - 1 = 0$. Describe the factorisation of the ideals generated by $2, 3, 5, 23$ in $\mathbf{Q}(\theta)$.

4. Marcus, Chapter 3, Exercise 30.