ON RUBIN'S VARIANT OF THE p-ADIC BIRCH AND SWINNERTON-DYER CONJECTURE II

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Abstract. Let $E/\mathbf{Q}$ be an elliptic curve with complex multiplication by the ring of integers of an imaginary quadratic field K. In 1991, by studying a certain special value of the Katz two-variable p-adic L-function lying outside the range of p-adic interpolation, K. Rubin formulated a p-adic variant of the Birch and Swinnerton-Dyer conjecture when $E(K)$ is infinite, and he proved that his conjecture is true for $E(K)$ of rank one.

When $E(K)$ is finite, however, the statement of Rubin's original conjecture no longer applies, and the relevant special value of the appropriate p-adic L-function is equal to zero. In this paper we extend our earlier work and give an unconditional proof of an analogue of Rubin's conjecture for the case in which $E(K)$ is finite.

1. Introduction

The goal of this article is to extend the results of [1] to give an unconditional proof of a certain variant of the p-adic Birch and Swinnerton-Dyer conjecture for elliptic curves with complex multiplication.

Let $E/\mathbf{Q}$ be an elliptic curve with complex multiplication by O_K, the ring of integers of an imaginary quadratic field K (this implies that K is necessarily of class number one). Let $p > 3$ be a prime of good, ordinary reduction for E; then we may write $pO_K = \mathfrak{p}\mathfrak{p}^*$, with $\mathfrak{p} = \pi O_K$ and $\mathfrak{p}^* = \pi^* O_K$.

Set $\mathcal{K}_\infty := K(E_{\pi^\infty})$, $\mathcal{K}^*_\infty := K(E_{\pi^{*\infty}})$, and $\mathfrak{K}_\infty := \mathcal{K}_\infty \mathcal{K}^*_\infty$. Let $\mathcal{O}$ denote the completion of the ring of integers of $\mathcal{K}^*_{\infty,\mathfrak{p}}$. For any extension L/K we set $\Lambda(L) := \Lambda(\mathrm{Gal}(L/K)) := \mathbf{Z}_p[[\mathrm{Gal}(L/K)]]$, and $\Lambda(L)_{\mathcal{O}} := \mathcal{O}[[\mathrm{Gal}(L/K)]]$.

Let

$$\psi : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_{\pi^\infty}) \xrightarrow{\sim} O^\times_{K,\mathfrak{p}} \xrightarrow{\sim} \mathbf{Z}_p^\times,$$
$$\psi^* : \mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}(E_{\pi^{*\infty}}) \xrightarrow{\sim} O^\times_{K,\mathfrak{p}^*} \xrightarrow{\sim} \mathbf{Z}_p^\times$$

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denote the natural $\mathbf{Z}_p^\times$-valued characters of $\mathrm{Gal}(\overline{K}/K)$ arising via Galois action on E_{π^∞} and $E_{\pi^{*\infty}}$ respectively. We may identify ψ with the Grossecharacter associated to E (and ψ^* with the complex conjugate $\overline{\psi}$ of this Grossencharacter), as described, for example, in [12, p. 325]. We write T and T^* for the $\mathfrak{p}$-adic and $\mathfrak{p}^*$-adic Tate modules of E respectively.

We now recall that the Katz two-variable p-adic L-function $\mathcal{L}_p \in \Lambda(\mathfrak{K}_\infty)_{\mathcal{O}}$ satisfies a p-adic interpolation formula that may be described as follows (see [12, Theorem 7.1] for the version given here, and also [5, Theorem II.4.14]). For all pairs of integers $j, k \in \mathbf{Z}$ with $0 \leq -j < k$, and for all characters $\chi : \mathrm{Gal}(K(E_p)/K) \to \overline{K}^\times$, we have

$$\mathcal{L}_\mathfrak{p}(\psi^k \psi^{*j} \chi) = A \cdot L(\psi^{-k} \overline{\psi}^{-j} \chi^{-1}, 0). \tag{1.1}$$

Here $L(\psi^{-k}\overline{\psi}^{-j}\chi^{-1}, s)$ denotes the complex Hecke L-function, and A denotes an explicit, non-zero factor whose precise description we shall not need.

Define

$$L_\mathfrak{p}(s) := \mathcal{L}_\mathfrak{p}(\psi < \psi >^{s-1}), \quad L_\mathfrak{p}^*(s) := \mathcal{L}_\mathfrak{p}(\psi^* < \psi^* >^{s-1})$$

for $s \in \mathbf{Z}_p$. The character ψ lies within the range of interpolation of $\mathcal{L}_\mathfrak{p}$, and the behaviour of $\mathcal{L}_\mathfrak{p}$ at ψ is predicted by the $\mathfrak{p}$-adic Birch and Swinnerton-Dyer conjecture for E (see [3, pages 133–134], [10, Theorem V.8]). Conjecturally, $\mathrm{ord}_{s=1} L_\mathfrak{p}(s)$ is equal to the rank r of $E(\mathbf{Q})$, and the exact value of $\lim_{s\to 1} L_\mathfrak{p}(s)/[s-1]^r$ may be decribed in terms of various arithmetic invariants associated to E.

The character ψ^*, however, lies outside the range of interpolation of $\mathcal{L}_\mathfrak{p}$, and the function $L_\mathfrak{p}^*(s)$ has not been studied nearly as much as $L_\mathfrak{p}(s)$. The first results concerning the behaviour of $L_\mathfrak{p}^*(s)$ were obtained by Karl Rubin (see [12], [13]). When $r \geq 1$, Rubin formulated a variant of the $\mathfrak{p}$-adic Birch and Swinnerton-Dyer conjecture for $L_\mathfrak{p}^*(s)$ which predicts that that $\mathrm{ord}_{s=1} L_\mathfrak{p}^*(s)$ is equal to $r-1$, and which gives a formula for $\lim_{s\to 1}[L_\mathfrak{p}^*(s)/(s-1)^{r-1}]$. Under suitable hypotheses, Rubin showed that his conjecture is equivalent to the usual $\mathfrak{p}$-adic Birch and Swinnerton-Dyer conjecture, and he proved both conjectures when $r = 1$. In the case $r = 1$, he then used these results to give the first examples of a p-adic construction of a global point of infinite order in $E(\mathbf{Q})$ directly from the special value of a p-adic L-function.

On the other hand, when $r = 0$, matters become quite different, and the results of [12], [13] do not apply. It is not hard to show that the functional equation satisfied by $\mathcal{L}_\mathfrak{p}$ (see [5, §6]) implies that $\mathrm{ord}_{s=1} L_\mathfrak{p}^*(s) \geq 1$. Rubin speculated that perhaps $\mathrm{ord}_{s=1} L_\mathfrak{p}^*(s) = 1$, and assuming that this was true, he also raised the question of determining the value of $\lim_{s\to 1}[L_\mathfrak{p}^*(s)/(s-1)]$ (see [13, Remark on p.74]).

In [1], we defined a *restricted Selmer group* $\check{\Sigma}_{\mathfrak{p}^*}(T^*) \subseteq H^1(K, T^*)$. This restricted Selmer group is defined by *reversing* the Selmer conditions above $\mathfrak{p}$ and $\mathfrak{p}^*$ that are used to define the usual Selmer group $\mathrm{Sel}(K, T^*)$. The $O_{K,\mathfrak{p}^*}$-module $\check{\Sigma}_{\mathfrak{p}^*}(K, T^*)$ is free of rank $|r-1|$, and if $r \geq 1$, then in fact $\check{\Sigma}_{\mathfrak{p}^*}(K, T^*) \subseteq \mathrm{Sel}(K, T^*)$. We also defined a similar group $\check{\Sigma}_{\mathfrak{p}}(K, T) \subseteq H^1(K, T)$, and we constructed a p-adic height pairing

$$[\,,\,]_{K,\mathfrak{p}^*} : \check{\Sigma}_{\mathfrak{p}}(K, T) \times \check{\Sigma}_{\mathfrak{p}^*}(K, T^*) \to O_{K,\mathfrak{p}^*}.$$

If $r \geq 1$, and if the $\mathfrak{p}^*$-adic Birch and Swinnerton-Dyer conjecture is true, then $[\,,\,]_{K,\mathfrak{p}^*}$ is non-degenerate. We conjectured that $[\,,\,]_{K,\mathfrak{p}^*}$ is also non-degenerate when $r = 0$.

It was shown in [1] that if $[\,,\,]_{K,\mathfrak{p}^*}$ is non-degenerate and the p-primary part of $\text{Ш}(E/K)$ is finite, then

$$\mathrm{ord}_{s=1} L_{\mathfrak{p}}^*(s) = \mathrm{rank}_{O_{K,\mathfrak{p}^*}}(\check{\Sigma}_{\mathfrak{p}^*}(K, T^*) = |r - 1|.$$

Under these assumptions, we also determined the value of $\lim_{s\to 1} L_{\mathfrak{p}}^*(s)/(s-1)^{|r-1|}$ up to multiplication by a p-adic unit (thereby recovering a weak form of [12, Corollary 11.3] in the case $r \geq 1$). In particular, our results implied that if $r = 0$ (in which case $\text{Ш}(E/K)$ is known to be finite; see [11]) and $[\,,\,]_{K,\mathfrak{p}^*}$ is non-degenerate, then $\mathrm{ord}_{s=1} L_{\mathfrak{p}}^*(s) = 1$, as was guessed by Rubin in [13].

Suppose now that $r = 0$. In this paper, we stregnthen the results of [1] by giving an unconditional proof of the fact that $\mathrm{ord}_{s=1}(L_{\mathfrak{p}}^*(s)) = 1$, and we also determine the exact value of the first derivative of $L_{\mathfrak{p}}^*(s)$ at $s = 1$. In order to state our main result, we must introduce some further notation.

Suppose that $y \in \check{\Sigma}_{\mathfrak{p}}(K, T)$ and $y^* \in \check{\Sigma}_{\mathfrak{p}^*}(K, T^*)$ are of infinite order, and let

$$\exp_{\mathfrak{p}}^* : H^1(K_{\mathfrak{p}}, T^*) \to \mathbf{Q}_p, \qquad \exp_{\mathfrak{p}^*}^* : H^1(K_{\mathfrak{p}^*}, T) \to \mathbf{Q}_p$$

denote the Bloch-Kato dual exponential maps. It may be shown that, via localisation, these induce maps (which we denote by the same symbols):

$$\exp_{\mathfrak{p}}^* : \check{\Sigma}_{\mathfrak{p}^*}(K, T^*) \to \mathbf{Q}_p, \qquad \exp_{\mathfrak{p}^*}^* : \check{\Sigma}_{\mathfrak{p}}(K, T) \to \mathbf{Q}_p.$$

Write $\mathfrak{f} \subseteq O_K$ for the conductor of the Grossencharacter associated to E, and let $\mathbf{N}(\mathfrak{f})$ denote the norm of this ideal. Set

$$\mathcal{L}_{\mathfrak{p}}'(\psi^*) := \lim_{s=1} \frac{L_{\mathfrak{p}}^*(s)}{s-1}.$$

The following result may perhaps be viewed as being an analogue of a similar exceptional zero phenomenon observed in the work of Mazur, Tate and Teitelbaum concerning p-adic Birch and Swinnerton-Dyer conjectures for elliptic curves *without* complex multiplication (see [6], [7]). It relates the *value* of $\mathcal{L}_{\mathfrak{p}}$ at a point within the range of p-adic interpolation to the *derivative* of $\mathcal{L}_{\mathfrak{p}}$ at a point lying outside the range of p-adic interpolation.

Theorem A. *Suppose that $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$.*

(a) The p-adic height pairing $[\,,\,]_{K,\mathfrak{p}^*}$ is non-degenerate and $\mathrm{ord}_{s=1} L_{\mathfrak{p}}^*(s) = 1$.

(b) We have that

$$(p-1)\cdot \left(1-\frac{1}{\psi(\mathfrak{p}^*)}\right)\cdot\left(1-\frac{1}{\psi^*(\mathfrak{p})}\right)\cdot\frac{\Omega_{\mathfrak{p}}\cdot\mathcal{L}'_{\mathfrak{p}}(\psi^*)}{[y,y^*]_{K,\mathfrak{p}^*}} =$$
$$\mathbf{N}(\mathfrak{f})\cdot\left(1-\frac{\psi^*(\mathfrak{p})}{p}\right)\cdot\left(1-\frac{\psi(\mathfrak{p}^*)}{p}\right)\cdot\frac{\mathcal{L}_{\mathfrak{p}^*}(\psi^*)}{\Omega_{\mathfrak{p}^*}\cdot\exp_{\mathfrak{p}}(y^*)\cdot\exp_{\mathfrak{p}^*}(y)},$$

where $\Omega_{\mathfrak{p}}$ and $\Omega_{\mathfrak{p}^*}$ are certain p-adic periods defined using the formal group associated to E (see Section 4 below).

It is interesting to compare Theorem A with [12, Theorem 10.1]. Both of these results are examples of the following more general phenomenon concerning certain special values of the Katz two-variable p-adic L-function $\mathcal{L}_{\mathfrak{p}}$. Let $k \geq 0$ be an integer, and set

$$\phi_k := \psi^{k+1}\psi^{*-k}, \qquad \phi_k^* := \psi^{-k}\psi^{*k+1}.$$

Then we see from (1.1) that ϕ_k lies within the range of interpolation of $\mathcal{L}_{\mathfrak{p}}$; for $k \geq 1$, the behaviour of $\mathcal{L}_{\mathfrak{p}}$ at ϕ_k is predicted by various conjectures due to Bloch, Beilinson, Kato and Perrin-Riou. On the other hand, the character ϕ_k^* lies outside the range of interpolation of $\mathcal{L}_{\mathfrak{p}}$, and as far the present author is aware, the behaviour of $\mathcal{L}_{\mathfrak{p}}$ at ϕ_k^* for $k \geq 1$ does not appear to have previously been studied. Using the techniques of [12], [1] and the present paper, it may be shown that the orders of vanishing of $\mathcal{L}_{\mathfrak{p}}$ at ϕ_k and ϕ_k^* are of opposite parity, and that the value of the first non-vanishing derivative of $\mathcal{L}_{\mathfrak{p}}$ in the ϕ_k-direction is related to the value of the first non-vanishing derivative of $\mathcal{L}_{\mathfrak{p}}$ in the ϕ_k^*-direction in a manner similar to [12, Theorem 10.1] (if $\mathcal{L}_{\mathfrak{p}}(\phi_k) = 0$) or to Theorem A above (if $\mathcal{L}_{\mathfrak{p}}(\phi_k) \neq 0$). We shall discuss this more fully in a future article (see [2]).

The strategy of the proof of Theorem A is similar to that employed in [12]; however, because we work with restricted Selmer groups rather than true Selmer groups, the details are rather different. The basic ideas may be described as follows. Using elliptic units, we construct canonical elements

$$s_{\mathfrak{p}} \in \check{\Sigma}_{\mathfrak{p}}(K,T), \qquad s_{\mathfrak{p}^*} \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*).$$

It follows from the construction that $s_{\mathfrak{p}^*}$ is of infinite order only if $\mathcal{L}'_{\mathfrak{p}}(\psi^*) \neq 0$. By analysing certain Kummer and cup product pairings, and using Wiles's explicit reciprocity for formal groups, we prove that

$$\exp_{\mathfrak{p}}^*(s_{\mathfrak{p}^*}) \doteq \mathcal{L}_{\mathfrak{p}}(\psi), \tag{1.2}$$

where the symbol "$\doteq$" denotes equality up to multiplication by a non-zero factor. Hence we see that if $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$, then $s_{\mathfrak{p}^*}$ is indeed of infinite order, and so $L_{\mathfrak{p}}^*(s)$ has a first-order zero at $s = 1$.

We then compute $[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*}$ using Kummer theory, Hilbert symbols, and Wiles's explicit reciprocity law, and we see that

$$[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*} \doteq \mathcal{L}_{\mathfrak{p}}(\psi) \cdot \mathcal{L}'_{\mathfrak{p}}(\psi^*) \tag{1.3}$$

The proof of Theorem A is then completed by showing that if $y \in \check{\Sigma}_{\mathfrak{p}}(K,T)$ and $y^* \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*)$ are both of infinite order, then

$$\frac{\exp^*_{\mathfrak{p}}(y^*) \cdot \exp^*_{\mathfrak{p}^*}(y)}{[y, y^*]_{K,\mathfrak{p}^*}} = \frac{\exp^*_{\mathfrak{p}}(s_{\mathfrak{p}^*}) \cdot \exp^*_{\mathfrak{p}^*}(s_{\mathfrak{p}})}{[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*}}$$

and so the desired result follows by applying (1.2) and (1.3).

An outline of the contents of this paper is as follows. In Section 2 we recall some basic properties of restricted Selmer groups, and we describe the construction of the canonical elements $s_{\mathfrak{p}}$ and $s_{\mathfrak{p}^*}$. In Section 3 we describe a local decomposition of the p-adic height pairing $[\,,\,]_{K,\mathfrak{p}^*}$ in terms of local Artin symbols. We recall some basic facts concerning the formal group $\hat{E}$ associated to E in Section 4, and we establish a number of conventions for use in subsequent calculations. We discuss properties of various Kummer pairings in Section 5, and we compute the value of $[\,,\,]_{K,\mathfrak{p}^*}$ on certain cohomology classes in restricted Selmer groups that are constructed using global units. In Section 6, we us the results of Section 5 to compute certain special values of the dual exponential map. Finally, in Section 7, we apply our previous results to the canonical elements $s_{\mathfrak{p}}$ and $s_{\mathfrak{p}^*}$, and we prove Theorem A.

Notation and conventions. Throughout this paper, K denotes an imaginary quadratic field of class number one. If L is any field, we write L^{ab} for the maximal abelian extension of L, and $\overline{L}$ for an algebraic closure of L.

For each integer $n \geq 1$, we write

$$\mathcal{K}_n := K(E_{\pi^n}), \quad \mathcal{K}^*_n := K(E_{\pi^{*n}}), \quad \mathfrak{K}_n := K(E_{p^n}) = \mathcal{K}_n \cdot \mathcal{K}^*_n,$$

and we write

$$\mathcal{K}_\infty := K(E_{\pi^\infty}), \quad \mathcal{K}^*_\infty := K(E_{\pi^{*\infty}}), \quad \mathfrak{K}_\infty := K(E_{p^\infty}).$$

We also put $\mathcal{N}_n := \mathcal{K}_n \cdot \mathcal{K}^*_\infty$, and we write $\mathfrak{m}_{n,\mathfrak{p}}$ for the maximal ideal in the completion of the ring of integers of $\mathcal{N}_{n,\mathfrak{p}}$.

We set $D_{\mathfrak{p}} := K_{\mathfrak{p}}/O_{K,\mathfrak{p}}$ and $D_{\mathfrak{p}^*} := K_{\mathfrak{p}^*}/O_{K,\mathfrak{p}^*}$.

For each integer $n \geq 1$, we let

$$e_n : E_{\pi^n} \times E_{\pi^{*n}} \to \mu_{p^n}$$

denote the Weil pairing, normalised as decribed in [8, §3.1.2] (the reader should note that this is *not* the same normalisation as that used in [12]). This pairing satisfies the identities

$$e_n(\pi^* \varsigma_n, \varsigma_n^*) = e_n(\varsigma_n, \pi \varsigma_n)$$

for $\varsigma_n \in E_{\pi^n}$, $\varsigma_n^* \in E_{\pi^{*n}}$, and

$$e_{n+1}(\varsigma_n, \varsigma_{n+1}^*) = e_n(\varsigma_n, \pi^* \varsigma_{n+1})$$

for $\varsigma_n \in E_{\pi^n}, \quad \varsigma_{n+1}^* \in E_{\pi^{*(n+1)}}$.

We set $W := E_{\pi^\infty}$ and $W^* := E_{\pi^{*\infty}}$. We write T and T^* for the π-adic and π^*-adic Tate modules of E respectively.

We use the following notation to denote various unit groups:

$$\begin{aligned}
&U_{n,\mathfrak{p}} := \text{units in } \mathcal{K}_{n,\mathfrak{p}} \text{ congruent to 1 modulo } \mathfrak{p};\\
&U_{n,\mathfrak{p}^*} := \text{units in } \mathcal{K}_{n,\mathfrak{p}^*} \text{ congruent to 1 modulo } \mathfrak{p}^*;\\
&U_{\infty,\mathfrak{p}} := \varprojlim U_{n,\mathfrak{p}}, \quad U_{\infty,\mathfrak{p}^*} := \varprojlim U_{n,\mathfrak{p}^*};\\
&U^*_{n,\mathfrak{p}} := \text{units in } \mathcal{K}^*_{n,\mathfrak{p}} \text{ congruent to 1 modulo } \mathfrak{p};\\
&U^*_{n,\mathfrak{p}^*} := \text{units in } \mathcal{K}^*_{n,\mathfrak{p}^*} \text{ congruent to 1 modulo } \mathfrak{p}^*;\\
&U^*_{\infty,\mathfrak{p}} := \varprojlim U^*_{n,\mathfrak{p}}, \quad U^*_{\infty,\mathfrak{p}^*} := \varprojlim U_{n,\mathfrak{p}^*},
\end{aligned}$$

where all inverse limits are taken with respect to the obvious norm maps. We also set

$$\begin{aligned}
&\mathcal{E}_n := \text{global units of } \mathcal{K}_n, \quad \mathcal{E}_n^* := \text{global units of } \mathcal{K}_n^*;\\
&\overline{\mathcal{E}}_n := \text{the closure of the projection of } \mathcal{E}_n \text{ into } U_{n,\mathfrak{p}};\\
&\overline{\mathcal{E}}^*_n := \text{the closure of the projection of } \mathcal{E}^*_n \text{ into } U^*_{n,\mathfrak{p}^*};\\
&\overline{\mathcal{E}}_\infty := \varprojlim \overline{\mathcal{E}}_n, \quad \overline{\mathcal{E}}^*_\infty := \varprojlim \overline{\mathcal{E}}^*_n.
\end{aligned}$$

Remark 1.1. Note that since the strong Leopoldt conjecture holds for all abelian extensions of K (see [4]), we have that

$$\overline{\mathcal{E}}_n \simeq \overline{\mathcal{E}}_n \otimes_{\mathbf{Z}} \mathbf{Z}_p, \quad \overline{\mathcal{E}}^*_n \simeq \overline{\mathcal{E}}^*_n \otimes_{\mathbf{Z}} \mathbf{Z}_p,$$

and so we may also view $\overline{\mathcal{E}}_\infty$ as being a submodule of $U_{\infty,\mathfrak{p}^*}$ and $\overline{\mathcal{E}}^*_\infty$ as being a submodule of $U^*_{\infty,\mathfrak{p}}$. We shall do this without further comment several times in what follows. □

Remark 1.2. It is important for the reader to bear in mind that every theorem or construction in this paper that depends upon a choice of prime $\mathfrak{p}$ of K lying above p also has a corresponding version in which the roles of $\mathfrak{p}$ and $\mathfrak{p}^*$ are interchanged. We shall sometimes make use of this fact without stating it explicitly. □

2. Restricted Selmer groups

In this section we shall recall some basic properties of restricted Selmer groups. We refer the reader to [1] for more details.

Suppose that F/K is any finite extension. For any place v of F, we define $H^1_f(F_v, W)$ to be the image of $E(F_v) \otimes D_{\mathfrak{p}}$ under the Kummer map

$$E(F_v) \otimes D_{\mathfrak{p}} \to H^1(F_v, W),$$

and we define $H^1_f(F_v, W^*)$ in a similar manner. Note that $H^1_f(F_v, W) = 0$ if $v \nmid \mathfrak{p}$. We also set

$$H^1_f(F_v, E_{\pi^n}) := \mathrm{Im}[E(F_v)/\pi^n E(F_v) \to H^1(F_v, E_{\pi^n})],$$
$$H^1_f(F_v, E_{\pi^{*n}}) := \mathrm{Im}[E(F_v)/\pi^{*n} E(F_v) \to H^1(F_v, E_{\pi^{*n}})].$$

Suppose that $M \in \{W, W^*, E_{\pi^n}, E_{\pi^{*n}}\}$ and that $\mathfrak{q} \in \{\mathfrak{p}, \mathfrak{p}^*\}$. If $c \in H^1(F, M)$, then we write $\mathrm{loc}_v(c)$ for the image of c in $H^1(F_v, M)$. We define

- the *true Selmer group* $\mathrm{Sel}(F, M)$ by

$$\mathrm{Sel}(F, M) = \left\{c \in H^1(F, M) \mid \mathrm{loc}_v(c) \in H^1_f(F_v, M) \,\text{for all}\, v\right\};$$

- the *relaxed Selmer group* $\mathrm{Sel}_{\mathrm{rel}}(F, M)$ by

$$\mathrm{Sel}_{\mathrm{rel}}(F, M) = \left\{c \in H^1(F, M) \mid \mathrm{loc}_v(c) \in H^1_f(F_v, M) \,\text{for all}\, v \text{ not dividing } p\right\};$$

- the $\mathfrak{q}$-*restricted Selmer group* (or simply *restricted Selmer group* for short when $\mathfrak{q}$ is understood) $\Sigma_{\mathfrak{q}}(F, M)$ by

$$\Sigma_{\mathfrak{q}}(F, M) = \{c \in \mathrm{Sel}_{\mathrm{rel}}(F, M) \mid \mathrm{loc}_v(c) = 0 \,\text{for all}\, v \text{ dividing } \mathfrak{q}\}.$$

(The terminology 'restricted Selmer group' is meant to reflect a choice of a combination of relaxed and strict Selmer conditions at places above p.)

If L/K is an infinite extension, we define

$$\Sigma_{\mathfrak{q}}(L, M) = \varinjlim \Sigma_{\mathfrak{q}}(L', M),$$

where the direct limits are taken with respect to restriction over all subfields $L' \subset L$ finite over K.

For any extension L/K, we set

$$\Sigma_{\mathfrak{q}}(L, M)^\wedge = X_{\mathfrak{q}}(L, M).$$

We record the following standard cohomological result that will be used later.

Lemma 2.1. *Let $n \geq 0$ be an integer, and suppose that L and M are fields with $K \subseteq L \subseteq M \subseteq \mathcal{N}_n$. Then, for every integer $m \geq 1$, the restriction maps*

$$H^1(L, E_{\pi^m}) \to H^1(M, E_{\pi^m}), \qquad H^1(L, \mu_{p^m}) \to H^1(M, \mu_{p^m})$$

are injective and they induce isomorphisms

$$H^1(L, E_{\pi^m}) \simeq H^1(M, E_{\pi^m})^{\mathrm{Gal}(M/L)}, \qquad H^1(L, \mu_{p^m}) \simeq H^1(M, \mu_{p^m})^{\mathrm{Gal}(M/L)}.$$

A similar result holds if L and M are replaced by $L_{\mathfrak{p}}$ and $M_{\mathfrak{p}}$ with $K_{\mathfrak{p}} \subseteq L_{\mathfrak{p}} \subseteq M_{\mathfrak{p}} \subseteq \mathcal{N}_{n,\mathfrak{p}}$.

Proof. This is quite standard, and may be proved via the argument given in [10, p.40], for example. □

We now explain how restricted Selmer groups may be described in terms of Kummer theory. Our starting point is the following result.

Lemma 2.2. *There is an isomorphism of $\mathrm{Gal}(\mathcal{K}_n^*/K)$-modules*

$$H^1(\mathcal{K}_n^*, E_{\pi^n}) \xrightarrow{\sim} \mathrm{Hom}(E_{\pi^{*n}}, \mathcal{K}_n^{*\times}/\mathcal{K}_n^{*\times p^n}); \quad f \mapsto \tilde{f}. \tag{2.1}$$

For each place v of $\mathcal{K}_n^*$, there is also a corresponding local isomorphism

$$H^1(\mathcal{K}_{n,v}^*, E_{\pi^n}) \xrightarrow{\sim} \mathrm{Hom}(E_{\pi^{*n}}, \mathcal{K}_{n,v}^{*\times}/\mathcal{K}_{n,v}^{*\times p^n}).$$

Proof. See [8, Lemme 3.8]. The isomorphism (2.1) is defined as follows. Let $f \in H^1(\mathcal{K}_n^*, E_{\pi^n})$, and recall that

$$e_n : E_{\pi^n} \times E_{\pi^{*n}} \to \mu_{p^n}$$

denotes the Weil pairing. We identify $\mathcal{K}_n^{*\times}/\mathcal{K}_n^{*\times p^n}$ with $H^1(\mathcal{K}_n^*, \mu_{p^n})$ via Kummer theory. If $\varsigma^* \in E_{\pi^{*n}}$, then $\tilde{f}(\varsigma^*) \in H^1(\mathcal{K}_n^*, \mu_{p^n})$ is defined to be the element represented by the cocycle

$$\sigma \mapsto e_n(f(\sigma), \varsigma^*)$$

for all $\sigma \in \mathrm{Gal}(\overline{K}/\mathcal{K}_n^*)$. □

Lemma 2.3. *For each place v of $\mathcal{K}_n^*$ with $v \nmid \mathfrak{p}^*$, there is an isomorphism*

$$E(\mathcal{K}_{n,v}^*)/\pi^n E(\mathcal{K}_{n,v}^*) \xrightarrow{\sim} \mathrm{Hom}(E_{\pi^{*n}}, O_{\mathcal{K}_{n,v}^*}^{\times}/O_{\mathcal{K}_{n,v}^*}^{\times p^n}).$$

Proof. See [8, Lemme 3.11]. □

Corollary 2.4. *Suppose that $h \in H^1(\mathcal{K}_n^*, E_{\pi^n})$. Then $h \in \Sigma_{\mathfrak{p}}(\mathcal{K}_n^*, E_{\pi^n})$ if and only if, for each $\varsigma \in E_{\pi^n}$, the following local conditions are satisfied:*

(a) $\tilde{h}(\varsigma) \in \mathcal{K}_{n,v}^{*\times p^n}$ for all $v \mid \mathfrak{p}$;

(b) $p^n \mid v_{\mathcal{K}_n^*}(\tilde{h}(\varsigma))$ for all $v \nmid \mathfrak{p}^*$.

(Note that we impose no local conditions at places lying above $\mathfrak{p}^*$.)

Proof. This follows directly from Lemmas 2.2 and 2.3. □

Proposition 2.5. *There are natural injections*

$$\rho : \mathrm{Hom}(T^*, (U^*_{\infty,\mathfrak{p}} \otimes \mathbf{Q})/\overline{\mathcal{E}}^*_\infty)^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)} \hookrightarrow \check{\Sigma}_{\mathfrak{p}}(K,T),$$
$$\rho^* : \mathrm{Hom}(T, (U_{\infty,\mathfrak{p}^*} \otimes \mathbf{Q})/\overline{\mathcal{E}}_\infty)^{\mathrm{Gal}(\mathcal{K}_\infty/K)} \hookrightarrow \check{\Sigma}_{\mathfrak{p}^*}(K,T^*)$$

Proof. The proof of this result is essentially the same, *mutatis mutandis*, as that of [12, Proposition 2.4]. The map ρ is defined as follows.

For any $f \in \mathrm{Hom}(T^*, (U^*_{\infty,\mathfrak{p}} \otimes \mathbf{Q})/\overline{\mathcal{E}}^*_\infty)^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)}$ and any integer $n \geq 1$, we define $f_n \in \mathrm{Hom}(E_{\pi^{*n}}, \mathcal{E}^*_n/\mathcal{E}^{*p^n}_n)^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)}$ to be the image of f under the following composition of maps:

$$\begin{aligned}\mathrm{Hom}(T^*, (U^*_{\infty,\mathfrak{p}} \otimes \mathbf{Q})/\overline{\mathcal{E}}^*_\infty)^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)} &\to \mathrm{Hom}(T^*, (U^*_{n,\mathfrak{p}} \otimes \mathbf{Q})/\overline{\mathcal{E}}^*_n)^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)}\\ &\to \mathrm{Hom}(E_{\pi^{*n}}, \mathcal{E}^*_n/\mathcal{E}^{*p^n}_n)^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)},\end{aligned}$$

where the first arrow is the map induced by the natural projection $U^*_{\infty,\mathfrak{p}} \to U^*_{n,\mathfrak{p}}$, and the second arrow is induced by raising to the p^n-th power in $U^*_{n,\mathfrak{p}}$.

Lemmas 2.1 and 2.2 yield an isomorphism

$$r_n : H^1(K, E_{\pi^n}) \xrightarrow{\sim} \mathrm{Hom}(E_{\pi^{*n}}, \mathcal{K}^{*\times}_n/\mathcal{K}^{*\times p^n}_n)^{\mathrm{Gal}(\mathcal{K}^*_n/K)} \tag{2.2}$$

(cf. [12, Lemma 2.1] or [8, Lemme 12], for instance). We define

$$\rho(f) := [(p-1)(\pi^*)^n r_n^{-1}(f_n)] \in \varprojlim_n H^1(K, E_{\pi^n}). \tag{2.3}$$

It follows from [8, Lemma 3.16] that $\rho(f)$ does indeed lie in $\varprojlim_n H^1(K, E_{\pi^n})$. It is not hard to check from the definition that ρ is injective. It follows from [1, Theorem 3.1, Proposition 3.2 and Corollary 3.3] that $r_n^{-1}(f_n) \in \Sigma_{\mathfrak{p}}(K, E_{\pi^n})$ if and only if the restriction of $r_n^{-1}(f_n)$ to $H^1(\mathfrak{K}_\infty, E_{\pi^n})$ is unramified outside $\mathfrak{p}^*$. It may be shown via an argument very similar to that given in [12, Lemmas 2.1 and 2.3] that this in fact the case. □

We shall now explain how elliptic units may be used (following the methods described in [12]) to construct canonical elements

$$s_{\mathfrak{p}} \in \check{\Sigma}_{\mathfrak{p}}(K,T), \quad s_{\mathfrak{p}^*} \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*)$$

when $r = 0$. These are the analogues in the present situation of the elements $x^{(1)}_{\mathfrak{p}} \in \check{\mathrm{Sel}}(K,T)$ and $x^{(1)}_{\mathfrak{p}^*} \in \check{\mathrm{Sel}}(K,T^*)$ constructed in [12] when $r = 1$.

Let $\mathcal{C}_\infty \subseteq \mathcal{E}_\infty$ and $\mathcal{C}^*_\infty \subseteq \mathcal{E}^*_\infty$ denote the norm-coherent systems of elliptic units constructed in [12, §3], and write $\overline{\mathcal{C}}_\infty$ and $\overline{\mathcal{C}}^*_\infty$ for the closures of $\mathcal{C}_\infty$ in $\overline{\mathcal{E}}_\infty$ and $\mathcal{C}^*_\infty$ in $\overline{\mathcal{E}}^*_\infty$ respectively. Set

$$\mathcal{I}^* := \mathrm{Ker}(\psi^* : \Lambda(\mathcal{K}^*_\infty) \to \mathbf{Z}_p), \quad \mathcal{I} := \mathrm{Ker}(\psi : \Lambda(\mathcal{K}_\infty) \to \mathbf{Z}_p),$$

and let ϑ^* be the generator of $\mathcal{I}^*$ fixed in [12, §6] (so $\vartheta^* = \gamma\psi^*(\gamma^{-1}) - 1$, where γ is any topological generator of $\mathrm{Gal}(\mathcal{K}_\infty^*/K)$ satisfying $\log_p(\psi^*(\gamma)) = p$). Recall that $\mathfrak{f} \subseteq O_K$ denotes the conductor of the Grossencharacter associated to E. Fix $B \in E_{\mathfrak{f}}/\mathrm{Gal}(\overline{K}/K)$ according to the recipe described in [12, §6] and let

$$\theta_B(w^*) \in \overline{\mathcal{C}}_\infty^* \subseteq U_{\infty,\mathfrak{p}}^* \otimes \mathbf{Q}$$

denote the elliptic unit constructed in [12, §3].

Now suppose that $r = 0$. Then $L_{\mathfrak{p}}(1) \neq 0$, and so we have (via [12, Theorem 7.2(i)], for example):

$$\overline{\mathcal{C}}_\infty \subseteq \overline{\mathcal{E}}_\infty \subset U_{\infty,\mathfrak{p}} \otimes \mathbf{Q} \quad \text{and} \quad \overline{\mathcal{C}}_\infty \nsubseteq \mathcal{I}(U_{\infty,\mathfrak{p}} \otimes \mathbf{Q}).$$

In particular, we have that $\overline{\mathcal{C}}_\infty \nsubseteq \mathcal{I}\overline{\mathcal{E}}_\infty \subseteq U_{\infty,\mathfrak{p}} \otimes \mathbf{Q}$. Similar remarks imply that also $\overline{\mathcal{C}}_\infty^* \nsubseteq \mathcal{I}^*\overline{\mathcal{E}}_\infty^* \subseteq U_{\infty,\mathfrak{p}^*}^* \otimes \mathbf{Q}$. Applying Remark 1.1, we deduce that

$$\overline{\mathcal{C}}_\infty^* \nsubseteq \mathcal{I}^*\overline{\mathcal{E}}_\infty^* \subseteq U_{\infty,\mathfrak{p}}^* \otimes \mathbf{Q}. \tag{2.4}$$

Also, if $L_{\mathfrak{p}}(1) \neq 0$, then this implies that $L_{\mathfrak{p}}^*(1) = 0$, and so from [12, Theorem 7.2(i)], it follows that we have

$$\overline{\mathcal{C}}_\infty^* \subseteq \mathcal{I}^*(U_{\infty,\mathfrak{p}}^* \otimes \mathbf{Q}). \tag{2.5}$$

Proposition 2.6. *There exists a unique homomorphism* $\sigma_{\mathfrak{p}} \in \mathrm{Hom}(T^*, (U_{\infty,\mathfrak{p}}^* \otimes \mathbf{Q})/\overline{\mathcal{E}}_\infty^*)^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)}$ *such that*

$$\sigma_{\mathfrak{p}}(w^*)^{\vartheta^*} = \theta_B(w^*)$$

in $\overline{\mathcal{E}}_\infty^*/\mathcal{I}^*\overline{\mathcal{E}}_\infty^*$.

Proof. It follows from [5, Chapter III, Proposition 1.3] that $U_{\infty,\mathfrak{p}}^*$ contains no ϑ^*-torsion elements. The existence of $\sigma_{\mathfrak{p}}$ therefore follows via an argument very similar to that of [12, Theorem 4.2]. □

Definition 2.7. We set

$$s_{\mathfrak{p}} := \rho(\sigma_{\mathfrak{p}}) \in \check{\Sigma}_{\mathfrak{p}}(K,T), \quad s_{\mathfrak{p}^*} := \rho^*(\sigma_{\mathfrak{p}^*}) \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*),$$

where of course the definition $\sigma_{\mathfrak{p}^*} \in \mathrm{Hom}(T, (U_{\infty,\mathfrak{p}^*} \otimes \mathbf{Q})/\overline{\mathcal{E}}_\infty)^{\mathrm{Gal}(\mathcal{K}_\infty/K)}$ the same, *mutatis mutandis*, as that of $\sigma_{\mathfrak{p}}$. □

Remark 2.8. It follows from the proof of Theorem 7.3(a) below that $s_{\mathfrak{p}^*}$ is of infinite order only if $\mathcal{L}_{\mathfrak{p}}'(\psi^*) \neq 0$. □

We conclude this section by recording the following result that we shall use in Sections 5 and 6.

Lemma 2.9. *There are injective homomorphisms*

$$\kappa : H^1(K, T) \to \varprojlim H^1(\mathcal{K}_n^*, \mathbf{Z}_p(1)) \tag{2.6}$$

$$\kappa^* : H^1(K, T^*) \to \varprojlim H^1(\mathcal{K}_n, \mathbf{Z}_p(1)), \tag{2.7}$$

where the inverse limits are taken with respect to the obvious corestriction maps. A similar result holds if K is replaced by $K_{\mathfrak{p}}$ or $K_{\mathfrak{p}^*}$.

Proof. We shall just explain the construction of κ^*, since that of κ is very similar.

Suppose that

$$c^* = [c_n^*] \in \varprojlim H^1(K_{\mathfrak{p}}, E_{\pi^{*n}}) \simeq H^1(K_{\mathfrak{p}}, T^*),$$

and consider the composition of maps

$$H^1(\mathcal{K}_{n+1,\mathfrak{p}}, \mu_{p^{n+1}}) \to H^1(\mathcal{K}_{n,\mathfrak{p}}, \mu_{p^{n+1}}) \to H^1(\mathcal{K}_{n,\mathfrak{p}}, \mu_{p^n}), \tag{2.8}$$

where the first arrow is given by corestriction and the second arrow is induced by the natural map $\mu_{p^{n+1}} \to \mu_{p^n}$. Just as in the case of (2.2), there is an isomorphism

$$r_n^* : H^1(K, E_{\pi^{*n}}) \xrightarrow{\sim} \mathrm{Hom}(E_{\pi^n}, \mathcal{K}_n^\times / \mathcal{K}_n^{\times p^n})^{\mathrm{Gal}(\mathcal{K}_n/K)}, \tag{2.9}$$

and it is not hard to check that (2.8) maps $\{r_{n+1}^*(c_{n+1}^*)\}(\pi^{*-(n+1)} w_{n+1})$ to $\{r_n^*(c_n^*)\}(\pi^{*-n} w_n)$. We may therefore define

$$\tilde{c^*}(w) := [\{r_n^*(c_n^*)\}(\pi^{*-n} w_n)] \in \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mu_{p^n}),$$

where the second inverse limit is taken with respect to the maps (2.8). It is a standard fact (see e.g. [15, Appendix B, Section B.3]) that

$$\varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mu_{p^n}) \simeq \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1)),$$

where the right-hand inverse limit is taken with respect to the obvious corestriction maps. We may therefore view $\tilde{c^*}(w)$ as being an element of $\varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1))$, and we write $\kappa^*(c^*) = [\kappa^*(c^*)_n] \in \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1))$ for this element. We remark that it follows from the construction of κ^* that , for each integer $n \geq 1$, we have:

$$\kappa^*(c^*)_n \equiv \{r_n^*(c_n^*)\}(\pi^{*-n} w_n) \pmod{\mathcal{K}_{n,\mathfrak{p}}^{\times p^n}} \tag{2.10}$$

in $H^1(\mathcal{K}_{n,\mathfrak{p}}, \mu_{p^n}) \simeq \mathcal{K}_{n,\mathfrak{p}}^\times / \mathcal{K}_{n,\mathfrak{p}}^{\times p^n}$. □

3. The p-adic height pairing on restricted Selmer groups

Our goal in this section is to describe a local decomposition of the p-adic height pairing

$$[\,,\,]_{K,\mathfrak{p}^*} : \check{\Sigma}_{\mathfrak{p}}(K,T) \times \check{\Sigma}_{\mathfrak{p}^*}(K,T^*) \to O_{K,\mathfrak{p}^*}.$$

in terms of Artin symbols; this is analogous to the local decomposition of the standard p-adic height pairing on true Selmer groups described in [8, Lemme 3.19].

We begin by recalling the outlines of the main steps in the construction of $[\,,\,]_{K,\mathfrak{p}^*}$. For more complete details, we refer the reader to [1, §4].

Let $\mathcal{Y}(\mathcal{K}_\infty^*)$ denote the Galois group over $\mathcal{K}_\infty^*$ of the maximal abelian pro-p extension of $\mathcal{K}_\infty^*$ that is unramified away from $\mathfrak{p}$ and totally split at all places of $\mathcal{K}_\infty^*$ lying above $\mathfrak{p}^*$. The first step in the construction of the pairing $[\,,\,]_{K,\mathfrak{p}^*}$ is the construction of an isomorphism

$$\Psi_K : \check{\Sigma}_{\mathfrak{p}}(K,T) \xrightarrow{\sim} \mathrm{Hom}(T^*, \mathcal{Y}(\mathcal{K}_\infty^*))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)}. \tag{3.1}$$

This isomorphism is constructed as follows. Write J_n for the group of finite ideles of $\mathcal{K}_n^*$, and let V_n denote the subgroup of J_n whose components are equal to 1 at all places dividing $\mathfrak{p}$ and are units at all places not dividing $\mathfrak{p}^*$. Set $C_n := J_n/(V_n \cdot \mathcal{K}_n^*)$, and write $C_n(p)$ for the p-primary subgroup of C_n.

Using Kummer theory (cf. Corollary 2.4 above), one constructs a surjective homomorphism

$$\eta_n : \mathrm{Hom}(E_{\pi^{*n}}, C_n)^{\mathrm{Gal}(\mathcal{K}_n^*/K)} \to \Sigma_{\mathfrak{p}}(K, E_{\pi^n})$$

with $|\,\mathrm{Ker}(\eta_n)|$ bounded independently of n. Let η_n' denote the map obtained from η_n via passage to the quotient by $\mathrm{Ker}(\eta_n)$. It may be shown that passing to inverse limits over the maps $\eta_n'^{-1}$ yields an isomorphism

$$\Xi_F : \varprojlim \Sigma_{\mathfrak{p}}(K, E_{\pi^n}) = \check{\Sigma}_{\mathfrak{p}}(K,T) \xrightarrow{\sim} \mathrm{Hom}(T^*, \varprojlim C_n(p))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)},$$

where the inverse limit $\varprojlim C_n(p)$ is taken with respect to the norm maps $\mathcal{K}_n^{*\times} \to \mathcal{K}_{n-1}^{*\times}$. One then shows via class field theory (along with the fact that the weak p-adic Leopoldt conjecture holds for K) that there is an isomorphism

$$\mathrm{Hom}(T^*, \varprojlim C_n(p))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)} \simeq \mathrm{Hom}(T^*, \mathcal{Y}(\mathcal{K}_\infty^*))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)}, \tag{3.2}$$

and composing this last isomorphism with with Ξ_K yields the desired isomorphism Ψ_K.

Next, by suitably interpreting restricted Selmer groups in terms of certain Galois groups (see [1, Theorem 3.1]), one shows that there is a natural homomorphism

$$\beta_K : \mathrm{Hom}(T^*, \mathcal{Y}(\mathcal{K}_\infty^*))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)} \to \mathrm{Hom}_{O_{K,\mathfrak{p}^*}}(\check{\Sigma}_{\mathfrak{p}^*}(K,T^*), O_{K,\mathfrak{p}^*}).$$

We thus obtain a map

$$\beta_K \circ \Psi_K : \check{\Sigma}_{\mathfrak{p}}(K,T) \to \mathrm{Hom}_{O_{K,\mathfrak{p}^*}}(\check{\Sigma}_{\mathfrak{p}^*}(K,T^*), O_{K,\mathfrak{p}^*}),$$

and this yields the p-adic height pairing pairing

$$[\,,\,]_{K,\mathfrak{p}^*} : \check{\Sigma}_{\mathfrak{p}}(K,T) \times \check{\Sigma}_{\mathfrak{p}^*}(K,T^*) \to O_{K,\mathfrak{p}^*}$$

on restricted Selmer groups.

In order to describe the local decomposition of $[\,,\,]_{K,\mathfrak{p}^*}$, we must introduce some further notation.

Suppose that

$$y = [y_n] \in \check{\Sigma}_{\mathfrak{p}}(K,T), \qquad y^* = [y_n^*] \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*).$$

For each positive integer n, we define q_n to be the map

$$q_n : \check{\Sigma}_{\mathfrak{p}}(K,T) \xrightarrow{\Psi_K} \mathrm{Hom}(T^*, \mathcal{Y}(\mathcal{K}_\infty^*))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)} \to \mathrm{Hom}(E_{\pi^{*n}}, C_n)^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)},$$

where the second arrow is the natural quotient map afforded by the isomorphism (3.2).

For each $\varsigma^* \in E_{\pi^{*n}}$, let $n(\varsigma^*)$ denote the exact power of π^* that kills ς^*. Let $S_{n,\varsigma^*}(y_n)$ denote any representative of $\eta_n^{-1}(y_n)(\varsigma^*)$ in J_n. For each finite place v of K, define $\{y,y^*\}_{n,v}^{(\varsigma^*)}$ to be the unique element of $O_K/\pi^{*n(\varsigma^*)}O_K$ such that

$$\{y,y^*\}_{n,v}^{(\varsigma^*)} \cdot \varsigma^* = y_n^*([S_{n,\varsigma^*}(y_n)_v, K_v^{\mathrm{ab}}/\mathcal{K}_{n,v}]),$$

where $[S_{n,\varsigma^*}(y_n)_v, K_v^{\mathrm{ab}}/\mathcal{K}_{n,v}] \in \mathrm{Gal}(K_v^{\mathrm{ab}}/K_{n,v})$ is the obvious local Artin symbol.

Proposition 3.1. *(cf.* [8, Lemma 3.19]*)*

(a) For any $\varsigma^* \in E_{\pi^{*n}}$, we have

$$[y,y^*]_{K,\mathfrak{p}^*} \cdot \varsigma^* = y_n^*([q_n(y)(\varsigma^*), K^{\mathrm{ab}}/\mathcal{K}_n^*]), \tag{3.3}$$

where $[q_n(y)(\varsigma^*), K^{\mathrm{ab}}/\mathcal{K}_n^*] \in \mathrm{Gal}(K^{\mathrm{ab}}/\mathcal{K}_n^*)$ is the obvious global Artin symbol.

(b) We have

$$[y,y^*]_{K,\mathfrak{p}^*} \equiv \sum_v \{y,y^*\}_{n,v}^{(\varsigma^*)} \pmod{\pi^{*n(\varsigma^*)}O_{K,\mathfrak{p}^*}}, \tag{3.4}$$

where the sum is over all finite places v of $\mathcal{K}_n^*$.

Proof. (a) This follows immediately from the following commutative diagram:

$$\begin{array}{ccccc}
\check{\Sigma}_{\mathfrak{p}}(K,T) & \xrightarrow{\Psi_K} & \mathrm{Hom}(T^*, \mathcal{Y}(\mathcal{K}_\infty^*))^{\mathrm{Gal}(\mathcal{K}_\infty^*/K)} & \xrightarrow{\beta_K} & \mathrm{Hom}_{O_{K,\mathfrak{p}^*}}(\check{\Sigma}_{\mathfrak{p}^*}(K,T^*), O_{K,\mathfrak{p}^*}) \\
\downarrow & & \downarrow & & \downarrow \\
\Sigma_{\mathfrak{p}}(K,E_{\pi^n}) & \longrightarrow & \mathrm{Hom}(E_{\pi^{*n}}, C_n)^{\mathrm{Gal}(K_\infty^*/K)} & \longrightarrow & \mathrm{Hom}(\Sigma_{\mathfrak{p}^*}(K,E_{\pi^{*n}}), O_K/\pi^{*n}O_K)
\end{array}$$

In this diagram, the second arrow on the bottom row is given by

$$f \mapsto (c \mapsto \{\varsigma^* \mapsto c([f(\varsigma^*), K^{\mathrm{ab}}/\mathcal{K}_n^*])\}),$$

where here we have canonically identified $O_K/\pi^{*n}O_K$ with $\mathrm{Hom}(E_{\pi^{*n}}, E_{\pi^{*n}})$ via the map

$$\beta \mapsto \{\varsigma^* \mapsto \beta \cdot \varsigma^*\}.$$

(b) This follows from the local decomposition of the global Artin symbol afforded via class field theory, viz. if $\alpha \in J_n$, then

$$[\alpha, K^{\mathrm{ab}}/\mathcal{K}_n^*] = \prod_v [\alpha_v, K_v^{\mathrm{ab}}/\mathcal{K}_{n,v}^*],$$

where the product is over all finite places v of $\mathcal{K}_n^*$. □

4. Formal Groups

The purpose of this section is to recall a number of facts concerning formal groups, and to establish certain conventions that we shall use later.

We fix a minimal Weierstrass model of $E/O_{K,\mathfrak{p}}$, and we write $\hat{E}$ for the associated formal group of E over $O_{K,\mathfrak{p}}$. Let $\hat{\mathbf{G}}_m$ denote the formal group over $O_{K,\mathfrak{p}}$ associated to the multiplicative group $\mathbf{G}_m$. If x is a point on $\mathbf{G}_m$ or on E, then we write $\hat{x}$ for the corresponding value of the parameter on $\hat{\mathbf{G}}_m$ or on $\hat{E}$. We denote the logarithm associated to $\hat{E}$ by $\lambda_{\hat{E}}(Z) = Z + (\text{higher order terms}) \in O_{K,\mathfrak{p}}[[Z]]$, and we write $\log_{E,\mathfrak{p}}$ for the corresponding $\mathfrak{p}$-adic logarithm associated to E. We denote the $\mathfrak{p}$-adic logarithm associated to $\mathbf{G}_m$ by $\log_{\mathfrak{p}}$.

Recall that $\mathcal{O}$ denotes the completion of the ring of integers of $\mathcal{K}_{\infty,\mathfrak{p}}^*$. Since $\hat{E}$ is a height one Lubin-Tate formal group, we may fix an isomorphism

$$\eta : \hat{\mathbf{G}}_m \to \hat{E}, \qquad \eta \in \mathcal{O}[[Z]].$$

As explained in [12, §6], this choice of isomorphism then yields:

(a) A generator $w^* = [w_n^*]$ of T^* such that for every $n \geq 1$ and $\varsigma \in E_{\pi^n}$, we have

$$\eta(\widehat{e_n(\pi^{*-n}\varsigma, w_n^*)}) = \hat{\varsigma}. \tag{4.1}$$

(b) A $\mathfrak{p}$-adic period $\Omega_{\mathfrak{p}} := \eta'(0) \in \mathcal{O}^\times$ which is such that

$$\Omega_{\mathfrak{p}}^\sigma = \psi^*(\sigma^{-1})\Omega_{\mathfrak{p}}$$

for every $\sigma \in \mathrm{Gal}(\overline{K}/K)$.

We fix a generator $w = [w_n]$ of T, and for each $n \geq 0$, we set

$$\zeta_n := e_n(\pi^{*-n}w_n, w_n^*);$$

so from (4.1) above, we have that

$$\eta(\hat{\zeta}_n) = \hat{w}_n.$$

We note that for each integer $n \geq 1$, our choice of η induces an isomorphism

$$\chi_n : \mu_{p^n} \xrightarrow{\sim} E_{\pi^n}; \qquad \zeta_n \mapsto w_n$$

which is $\mathrm{Gal}(\overline{K}/\mathcal{K}_n^*)$-equivariant; this in turn induces an isomorphism (which we denote by the same symbol)

$$\chi_n : H^1(\mathcal{N}_{n,\mathfrak{p}}, \mu_{p^n}) \xrightarrow{\sim} H^1(\mathcal{N}_{n,\mathfrak{p}}, E_{\pi^n}). \tag{4.2}$$

Proposition 4.1. *Recall that for each integer $n \geq 1$, $\mathfrak{m}_{n,\mathfrak{p}}$ denotes the maximal ideal in the completion of the ring of integers of $\mathcal{N}_{n,\mathfrak{p}}$.*

(a) With notation as above, the following diagram commutes:

$$\begin{array}{ccc} H^1(\mathcal{N}_{n,\mathfrak{p}}, \mu_{p^n}) & \xrightarrow[\sim]{\chi_n} & H^1(\mathcal{N}_{n,\mathfrak{p}}, E_{\pi^n}) \\ \uparrow & & \uparrow \\ \dfrac{\hat{\mathbf{G}}_m(\mathfrak{m}_{n,\mathfrak{p}})}{\hat{\mathbf{G}}_m(\mathfrak{m}_{n,\mathfrak{p}})^{p^n}} \simeq \dfrac{O^{\times}_{\mathcal{N}_{n,\mathfrak{p}}}}{O^{\times p^n}_{\mathcal{N}_{n,\mathfrak{p}}}} & \xrightarrow[\sim]{\eta} & \dfrac{\hat{E}(\mathfrak{m}_{n,\mathfrak{p}})}{\mathfrak{p}^n \hat{E}(\mathfrak{m}_{n,\mathfrak{p}})}. \end{array}$$

(Here the vertical arrows denote the natural maps afforded by Kummer theory on $\hat{\mathbf{G}}_m$ and $\hat{E}$.)

(b) If $\hat{x} \in \hat{\mathbf{G}}_m(\mathfrak{m}_{n,\mathfrak{p}})$, then

$$\log_{E,\mathfrak{p}}(\eta(\hat{x})) \equiv \Omega_{\mathfrak{p}} \cdot \log_{\mathfrak{p}}(\hat{x}) \pmod{\mathfrak{m}_{n,\mathfrak{p}}^{p^n}}$$

on $\hat{\mathbf{G}}_m(\mathfrak{m}_{n,\mathfrak{p}})/\hat{\mathbf{G}}_m(\mathfrak{m}_{n,\mathfrak{p}})^{p^n}$.

Proof. (a) This follows directly from the definitions of η and χ_n.

(b) See the proof of [12, Corollary 9.2]. □

5. Kummer pairings and p-adic heights

In this section we shall compute the value of the p-adic height pairing $[\,,\,]_{K,\mathfrak{p}}$ on elements of restricted Selmer groups that are constructed via Proposition 2.5. This is accomplished by using Proposition 3.1 to express these values in terms of certain Kummer pairings, and then applying Wiles's explicit reciprocity law.

Definition 5.1. For each integer $n \geq 1$, we define a pairing

$$(\,,\,)_{\mathfrak{p},E_{\pi^{*n}}} : U^*_{n,\mathfrak{p}} \times H^1(K_{\mathfrak{p}}, E_{\pi^{*n}}) \to E_{\pi^{*n}} \tag{5.1}$$

by

$$(u_n, c_n^*)_{\mathfrak{p},E_{\pi^{*n}}} = c_n^*([u_n, K_{\mathfrak{p}}^{\mathrm{ab}}/\mathcal{K}^*_{n,\mathfrak{p}}]) := \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c_n^*) \cdot w_n^*,$$

with $\alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c_n^*) \in O_K/\mathfrak{p}^{*n}O_K \simeq \mathbf{Z}_p/p^n\mathbf{Z}_p$.

The pairings (5.1) give a pairing

$$(\, , \,)_{\mathfrak{p},T^*} : U^*_{\infty,\mathfrak{p}} \times H^1(K_{\mathfrak{p}}, T^*) \to T^*; \quad (u, c^*) \mapsto \alpha_{\mathfrak{p},T^*}(u, c^*) \cdot w^* \tag{5.2}$$

that is defined as follows. Suppose that $u = [u_n] \in U^*_{\infty,\mathfrak{p}}$ and

$$c^* = [c_n^*] \in \varprojlim H^1(K_{\mathfrak{p}}, E_{\pi^{*n}}) \simeq H^1(K_{\mathfrak{p}}, T^*).$$

Then we set

$$(u, c^*)_{\mathfrak{p},T^*} = [(u_n, c_n^*)_{\mathfrak{p},E_{\pi^{*n}}}] = [\alpha_{\mathfrak{p},E_{\pi^{*n}}}] \cdot w^* := \alpha_{\mathfrak{p},T^*}(u, c^*) \cdot w^*.$$

□

Recall from Proposition 2.5 above that there is a natural injection

$$\rho : \mathrm{Hom}(T^*, (U^*_{\infty,\mathfrak{p}} \otimes \mathbf{Q})/\overline{\mathcal{E}}^*_{\infty})^{\mathrm{Gal}(\mathcal{K}^*_{\infty}/K)} \hookrightarrow \check{\Sigma}_{\mathfrak{p}}(K, T).$$

Proposition 5.2. *Suppose that*

$$\xi \in \mathrm{Hom}(T^*, U^*_{\infty,\mathfrak{p}}/\overline{\mathcal{E}}^*_{\infty})^{\mathrm{Gal}(\mathcal{K}^*_{\infty}/K)}.$$

Let $\overline{\xi(w^*)} \in U^*_{\infty,\mathfrak{p}}$ *denote any lift of* $\xi(w^*)$, *and suppose that* $y^* \in \check{\Sigma}_{\mathfrak{p}^*}(K, T^*)$. *Then we have*

$$[\rho(\xi), y^*]_{K,\mathfrak{p}^*} \cdot w^* = (\overline{\xi(w^*)}), \mathrm{loc}_{\mathfrak{p}}(y^*))_{\mathfrak{p},T^*},$$

and so it follows that

$$[\rho(\xi), y^*]_{K,\mathfrak{p}^*} = [\alpha_{\mathfrak{p},E_{\pi^{*n}}}((\overline{\xi(w^*)})_n, \mathrm{loc}_{\mathfrak{p}}(y_n^*))] = \alpha_{\mathfrak{p},T^*}(\overline{\xi(w^*)}, \mathrm{loc}_{\mathfrak{p}}(y^*)).$$

Proof. This follows directly from Proposition 3.1 and Definition 5.1. □

We shall now explain how the Kummer pairings (5.1) and (5.2) may be expressed in terms of certain Hilbert symbols.

Definition 5.3. We define the Hilbert pairing

$$(\, , \,)_{\mathfrak{p},\mu_{p^n}} : \frac{\mathcal{N}^{\times}_{n,\mathfrak{p}}}{\mathcal{N}^{\times p^n}_{n,\mathfrak{p}}} \times \frac{\mathcal{N}^{\times}_{n,\mathfrak{p}}}{\mathcal{N}^{\times p^n}_{n,\mathfrak{p}}} \to \mu_{p^n}$$

by

$$(a, b)_{\mathfrak{p},\mu_{p^n}} = \frac{(b^{1/p^n})^{\sigma_a}}{b^{1/p^n}},$$

where σ_a denotes the local Artin symbol $[a, K^{\mathrm{ab}}_{\mathfrak{p}}/\mathcal{N}_{n,\mathfrak{p}}]$ and b^{1/p^n} is any p^n-th root of b in $K^{\mathrm{ab}}_{\mathfrak{p}}$.

We remark that it is a standard property of the Hilbert pairing (see e.g. [17, Chapter XIV, §2]) that

$$(a,b)_{\mathfrak{p},\mu_{p^n}} = (b,a)^{-1}_{\mathfrak{p},\mu_{p^n}}. \tag{5.3}$$

□

Lemma 5.4. *Suppose that $u \in U^*_{\infty,\mathfrak{p}}$ and $c^* \in H^1(K_\mathfrak{p}, T^*)$. Then*

$$(u_n, r^*_n(c^*_n)(\pi^{*-n}w_n))_{\mathfrak{p},\mu_{p^n}} = \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n)\cdot\zeta_n.$$

Proof. It follows from the definition of r^*_n (see (2.9)) that

$$\begin{aligned}(u_n, r^*_n(c^*_n)(\pi^{*-n}w_n))_{\mathfrak{p},\mu_{p^n}} &= e_n(\pi^{*-n}w_n, c^*_n([u_n, K^{\mathrm{ab}}_\mathfrak{p}/\mathcal{N}_{n,\mathfrak{p}}]) \\ &= e_n(\pi^{*-n}w_n, c^*_n([u_n, K^{\mathrm{ab}}_\mathfrak{p}/\mathcal{K}^*_{n,\mathfrak{p}}]) \quad (\text{since } u_n \in \mathcal{K}^*_{n,\mathfrak{p}}) \\ &= e_n(\pi^{*-n}w_n, \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n)\cdot w^*_n) \\ &= \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n)\cdot e_n(w_n, \pi^{-n}w^*_n) \\ &= \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n)\cdot\zeta_n,\end{aligned}$$

as claimed. □

Definition 5.5. For each integer $n \geq 1$, we define a pairing

$$(\ ,\ )_{\mathfrak{p},E_{\pi^n}} : \mathcal{N}^\times_{n,\mathfrak{p}} \times H^1(\mathcal{N}_{n,\mathfrak{p}}, E_{\pi^n}) \to E_{\pi^n}$$

by

$$(\nu, y)_{\mathfrak{p},E_{\pi^n}} = y([\nu, K^{\mathrm{ab}}_\mathfrak{p}/\mathcal{N}_{n,\mathfrak{p}}]).$$

It is not hard to check (using Lemma 2.1) that the pairings $(\ ,\ )_{\mathfrak{p},E_{\pi^n}}$ combine to yield a pairing

$$(\ ,\ )_{\mathfrak{p},T} : U_{\infty,\mathfrak{p}} \times H^1(K_\mathfrak{p}, T) \to T; \quad (u,c) \mapsto \alpha_{\mathfrak{p},T}(u,c)\cdot w$$

that is defined as follows. Suppose that $u = [u_n] \in U_{\infty,\mathfrak{p}}$, and that

$$c = [c_n] \in \varprojlim H^1(K_\mathfrak{p}, E_{\pi^n}) \simeq H^1(K_\mathfrak{p}, T).$$

The we set

$$(u,c)_{\mathfrak{p},T} = [(u_n, c_n)_{\mathfrak{p},E_{\pi^n}}] = [\alpha_{\mathfrak{p},E_{\pi^n}}(u_n, c_n)]\cdot w := \alpha_{\mathfrak{p},T}(u,c)\cdot w. \quad \square$$

Lemma 5.6. *(a) Suppose that $c \in H^1(K_\mathfrak{p}, T)$, and that $u \in U_{\infty,\mathfrak{p}}$. Then*

$$(u_n, r_n(c_n)(\pi^{-n}w^*_n))_{\mathfrak{p},\mu_{p^n}} = \alpha_{\mathfrak{p},E_{\pi^n}}\cdot\zeta_n.$$

(b) Suppose that $u \in U^*_{\infty,\mathfrak{p}}$ and $c^* \in H^1(K_\mathfrak{p}, T^*)$. Then

$$(u_n, \chi_n(r^*_n(c^*_n(\pi^{*-n}w_n)))_{\mathfrak{p},E_{\pi^n}} = -(r^*_n(c^*_n)(\pi^{*-n}w_n), \chi_n(u_n))_{\mathfrak{p},E_{\pi^n}} = \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c_n)\cdot w_n.$$

(Recall that the isomorphism χ_n is defined in (4.2).*)*

Proof. Part (a) may be proved in exactly the same way as Lemma 5.4, while Part (b) is an immediate consequence of the same lemma. □

Suppose now that $u \in U^*_{\infty,\mathfrak{p}}$ and that $c^* \in H^1(K_{\mathfrak{p}}, T^*)$ with $\kappa^*(c^*) \in U_{\infty,\mathfrak{p}} \subseteq \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1))$. Lemma 5.6(b) implies that, for each integer $n \geq 1$, $\alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n)$ may be determined by calculating the value of $(\kappa^*(c^*)_n, \chi_n(u_n))_{\mathfrak{p},E_{\mathfrak{p}^n}}$; this in turn may be done by using Wiles's explicit reciprocity law applied to the formal group $\hat{E}$ associated to E, as we shall now describe.

Let $g_{\kappa^*(c^*),w}(Z) \in O_{K,\mathfrak{p}}[[Z]]$ denote the Coleman power series of $\kappa^*(c^*) \in U_{\infty,\mathfrak{p}}$; so, for every integer $n > 0$, we have

$$g_{\kappa^*(c^*),w}(\hat{w}_n) = \kappa^*(c^*)_n.$$

Set

$$\delta g_{\kappa^*(c^*),w}(Z) := \frac{g'_{\kappa^*(c^*),w}(Z)}{g_{\kappa^*(c^*),w}(Z) \cdot \lambda'_E(Z)},$$

and write

$$\delta_w(\kappa^*(c^*)) = \delta g_{\kappa^*(c^*),w}(0) := \frac{g'_{\kappa^*(c^*),w}(0)}{g_{\kappa^*(c^*),w}(0) \cdot \lambda'_E(0)} = \frac{g'_{\kappa^*(c^*),w}(0)}{g_{\kappa^*(c^*),w}(0)},$$

(where for the last equality we have used the fact that $\lambda_{\widehat{E}}(Z) = Z +$ (higher order terms)).

Proposition 5.7. *Suppose that $u \in U^*_{\infty,\mathfrak{p}}$ and that $c^* \in H^1(K_{\mathfrak{p}}, T^*)$ with $\kappa^*(c^*) \in U_{\infty,\mathfrak{p}} \subseteq \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1))$. Then, with notation as above, we have*

$$\begin{aligned} \alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n) &\equiv \left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \times \delta_w(\kappa^*(c^*)) \\ &\quad \times \Omega_{\mathfrak{p}} \times \sum_{\sigma \in \mathrm{Gal}(\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}})} \psi^*(\sigma^{-1}) \log_{\mathfrak{p}}(u_n^{\sigma}) \pmod{\mathfrak{p}^n}, \end{aligned} \tag{5.4}$$

where $\Omega_{\mathfrak{p}} \in \mathcal{O}^{\times}$ is the $\mathfrak{p}$-adic period described in Section 4.

Hence we have $(u, c^*)_{\mathfrak{p},T^*} = \alpha_{\mathfrak{p},T^*}(u, c^*) \cdot w^*$, where

$$\alpha_{\mathfrak{p},T^*}(u, c^*) = \left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \times \delta_w(\kappa^*(c^*)) \times \Omega_{\mathfrak{p}} \times \lim_{n\to\infty} \left\{ \sum_{\sigma \in \mathrm{Gal}(\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}})} \psi^*(\sigma^{-1}) \log_{\mathfrak{p}}(u_n^{\sigma}) \right\}. \tag{5.5}$$

Proof. Applying Wiles's explicit reciprocity law (see [5, Chapter I, Theorem 4.2]) to evaluate $(\kappa^*(c^*)_n, \chi_n(u_n))_{\mathfrak{p},E_{\pi^n}}$, we see (using Lemma 5.6) that the value of $\alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n)$ is given by:

$$\alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n) \equiv -\,\mathrm{Tr}_{\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}}} \left\{ \pi^{-n}\, \mathrm{Tr}_{\mathfrak{K}_{n,\mathfrak{p}}/\mathcal{K}^*_{n,\mathfrak{p}}} \left(\log_{E,\mathfrak{p}}(\widehat{\chi_n(u_n)})) \cdot \delta g_{\kappa^*(c^*),w}(\hat{w}_n) \right) \right\} \pmod{\mathfrak{p}^n}. \tag{5.6}$$

Since $u_n \in U^*_{n,\mathfrak{p}}$ and χ_n is $\mathrm{Gal}(\overline{K}/\mathcal{K}^*_n)$-equivariant, it follows that $\log_{E,\mathfrak{p}}(\widehat{\chi_n(u_n)}) \in \mathcal{K}^*_{n,\mathfrak{p}}$. Also, we have that $\pi^{-n}\operatorname{Tr}_{\mathfrak{K}_{n,\mathfrak{p}}/\mathcal{K}^*_{n,\mathfrak{p}}}(\delta g_{\kappa^*(c^*),w}(\hat{w}_n)) \in K_{\mathfrak{p}}$ because $\hat{w}_n \in \mathcal{K}_{n,\mathfrak{p}}$. Hence (5.6) implies that

$$\alpha_{\mathfrak{p},E_{\pi^{*n}}}(u_n, c^*_n) \equiv -\left\{\pi^{-n}\operatorname{Tr}_{\mathfrak{K}_{n,\mathfrak{p}}/\mathcal{K}^*_{n,\mathfrak{p}}}(\delta g_{\kappa^*(c^*)}(\hat{w}_n))\right\} \cdot \operatorname{Tr}_{\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}}}(\log_{E,\mathfrak{p}}(\widehat{\chi_n(u_n)})) \pmod{\mathfrak{p}^n}. \tag{5.7}$$

From Proposition 4.1(b), we see that

$$\log_{E,\mathfrak{p}}(\widehat{\chi_n(u_n)}) \equiv \Omega_{\mathfrak{p}} \cdot \log_{\mathfrak{p}}(u_n) \pmod{\mathfrak{p}^n},$$

and this implies that

$$\operatorname{Tr}_{\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}}}(\log_{E,\mathfrak{p}}(\widehat{\chi_n(u_n)})) \equiv \Omega_{\mathfrak{p}} \sum_{\sigma\in\mathrm{Gal}(\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}})} \psi^*(\sigma^{-1}\log_{\mathfrak{p}}(u_n^{\sigma})) \pmod{\mathfrak{p}^{n+1}}. \tag{5.8}$$

We also observe that [5, Chapter II, Proposition 4.5(iii)] implies that

$$\pi^{-n}\operatorname{Tr}_{\mathfrak{K}_{n,\mathfrak{p}}/\mathcal{K}^*_{n,\mathfrak{p}}}(\delta g_{\kappa^*(c^*),w}(\hat{w}_n)) = \left(1 - \frac{\psi^*(\mathfrak{p})}{p}\right) \times \delta_w(\kappa^*(c^*)), \tag{5.9}$$

which is independent of n. The congruence (5.4) now follows from (5.7), (5.8) and (5.9). The equality (5.5) is a direct consequence of (5.4). □

Suppose now that

$$\xi \in \mathrm{Hom}(T^*, U^*_{\infty,\mathfrak{p}}/\overline{\mathcal{E}^*_\infty})^{\mathrm{Gal}(\mathcal{K}^*_\infty/K)}, \quad \xi^* \in \mathrm{Hom}(T, U_{\infty,\mathfrak{p}^*}/\overline{\mathcal{E}_\infty})^{\mathrm{Gal}(\mathcal{K}_\infty/K)},$$

and set

$$c := \rho(\xi) \in \check{\Sigma}_{\mathfrak{p}}(K,T), \quad c^* := \rho^*(\xi^*) \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*).$$

Then $c^* = [c^*_n] = [(p-1)\pi^n r_n^{*-1}(\xi^*_n)] \in \varprojlim_n H^1(K, E_{\pi^n})$. For any $\varsigma \in E_{\pi^n}$, a routine computation shows that

$$r^*_n(c^*_n)(\varsigma) = \xi^*_n((p-1)\pi^{*n}\varsigma),$$

and so setting $\varsigma = \pi^{*-n}w_n$, we see that

$$r^*_n(c^*_n)(\pi^{*-n}w_n) = \xi^*_n(w_n)^{p-1}.$$

Hence it follows from (2.10) that

$$\kappa^*(c^*)_n \equiv \xi^*_n(w_n)^{p-1} \pmod{\mathcal{K}^{\times p^n}_{n,\mathfrak{p}}} \tag{5.10}$$

in $H^1(\mathcal{K}_{n,\mathfrak{p}}, \mu_{p^n}) \simeq \mathcal{K}^\times_{n,\mathfrak{p}}/\mathcal{K}^{\times p^n}_{n,\mathfrak{p}}$.

We now observe that as ξ^* is $\mathrm{Gal}(\mathcal{K}_\infty/K)$-equivariant, it follows that $\xi^*(w)^{\vartheta} \in \overline{\mathcal{E}}_\infty/\mathcal{I}\overline{\mathcal{E}}_\infty$. The following result describes the relationship between $\kappa^*(c^*)$ and $\xi^*(w)^{\vartheta}$.

Proposition 5.8. *With the above notation, we have*

$$\delta_w(\kappa^*(c^*)) = \delta_w(\xi^*(w)^{\vartheta}),$$

where $\delta_w(\xi^*(w)^{\vartheta})$ denotes δ_w applied to any lift in $\overline{\mathcal{E}}_\infty$ of $\xi^*(w)^{\vartheta} \in \overline{\mathcal{E}}_\infty/\mathcal{I}\overline{\mathcal{E}}_\infty$.

Proof. Suppose that $\xi^*(w) = \alpha = [\alpha_n]$, with $\alpha_n \in U_{n,\mathfrak{p}^*}/\overline{\mathcal{E}}_n$, and set $\xi^*(w) = \beta = [\beta_n]$, with $\beta_n \in \overline{\mathcal{E}}_n/\mathcal{I}\overline{\mathcal{E}}_n$. It follows from the definition of ξ_n^* (see Proposition 2.5) that

$$\xi_n^*(w_n) = \alpha^{p^n} \in \overline{\mathcal{E}}_n/\overline{\mathcal{E}}_n^{p^n} = \mathcal{E}_n/\mathcal{E}_n^{p^n}.$$

For each $n \geq 1$, let ϑ_n denote the projection of ϑ to $\mathbf{Z}_p[\mathrm{Gal}(\mathcal{K}_n/K)]$. It is shown in [12, Lemma 6.3] that

$$\vartheta_n \sum_{\sigma \in \mathrm{Gal}(\mathcal{K}_n/K)} \psi^{-1}(\overline{\sigma})\sigma \equiv -(p-1)p^n \pmod{p^n \mathcal{I}\mathbf{Z}_p[\mathrm{Gal}(\mathcal{K}_n/K)]},$$

where $\overline{\sigma}$ denotes any lift of $\sigma \in \mathrm{Gal}(\mathcal{K}_n/K)$ to $\mathrm{Gal}(\mathcal{K}_\infty/K)$.

Hence we have (using additive notation for Galois action):

$$\begin{aligned}\vartheta\xi_n^*(w_n) &= p^n\beta_n \\ &\equiv -(p-1)^{-1}\vartheta\left(\sum_{\sigma \in \mathrm{Gal}(\mathcal{K}_n/K)} \psi^{-1}(\overline{\sigma})\sigma\right)\beta_n \pmod{p^n\vartheta\overline{\mathcal{E}}_n}.\end{aligned}$$

Since $\overline{\mathcal{E}}_n$ has no ϑ-torsion (see [5, Chapter III, Proposition 1.3]), we can divide this relation by ϑ and apply (5.10) to obtain

$$\begin{aligned}(p-1)\xi_n^*(w_n) &\equiv \left(\sum_{\sigma \in \mathrm{Gal}(\mathcal{K}_n/K)} \psi^{-1}(\overline{\sigma})\sigma\right)\beta_n \pmod{p^n\overline{\mathcal{E}}_n} \\ &\equiv \kappa^*(c^*)_n \pmod{p^n\overline{\mathcal{E}}_n}.\end{aligned}$$

It now follows from [5, Chapter II, Lemma 4.8] that

$$\delta_w(\kappa^*(c^*) = \delta_w(\beta),$$

and this implies the desired result. $\square$

Let $\overline{\xi(w^*)} \in U^*_{\infty,\mathfrak{p}}$ be any lift of $\xi(w^*)$.

Theorem 5.9. *(a) For any $y^* \in \check{\Sigma}_{\mathfrak{p}^*}(K, T^*)$, we have*

$$[\rho(\xi), y^*]_{K,\mathfrak{p}^*} = (p-1)\left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \times \delta_w(\kappa^*(y^*)) \times \Omega_{\mathfrak{p}} \times \lim_{n\to\infty}\left\{\sum_{\sigma \in \mathrm{Gal}(\mathcal{K}^*_{n,\mathfrak{p}}/K_{\mathfrak{p}})} \psi^*(\sigma^{-1}) \log_{\mathfrak{p}}((\overline{\xi(w^*)})_n^{\sigma})\right\}.$$

(b) We have

$$[\rho(\xi), \rho^*(\xi^*)]_{K,\mathfrak{p}^*} = (p-1)\left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \times \delta_w(\xi^*(w)^\vartheta) \times \Omega_\mathfrak{p} \times \lim_{n\to\infty} \left\{ \sum_{\sigma \in \mathrm{Gal}(\mathcal{K}^*_{n,\mathfrak{p}}/K_\mathfrak{p})} \psi^*(\sigma^{-1}) \log_\mathfrak{p}((\overline{\xi(w^*)})^\sigma_n) \right\}.$$

Proof. This result follows directly from Propositions 5.2, 5.7 and 5.8. □

6. The Dual Exponential Map

Suppose that $c^* \in H^1(K_\mathfrak{p}, T^*)$ with $\kappa^*(c^*) \in U_{\infty,\mathfrak{p}} \subseteq \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1))$, and let

$$\exp^*_\mathfrak{p} : H^1(K_\mathfrak{p}, T^*) \to \mathbf{Q}_p$$

denote the Bloch-Kato dual exponential map (see e.g. [16, Section 5] for an account of the dual exponential map associated to an elliptic curve). In this section we shall evaluate $\exp^*_\mathfrak{p}(c^*)$ in terms of $\kappa^*(c^*)$.

We begin by describing the relationship between the cup product pairing

$$\cup : H^1(K_\mathfrak{p}, T) \times H^1(K_\mathfrak{p}, T^*) \to \mathbf{Z}_p \tag{6.1}$$

and the Kummer pairing $(\ ,\ )_{\mathfrak{p},T^*}$ of the previous section. This result is probably well known, but we are not aware of a written reference, and so we include a proof here.

Proposition 6.1. *Suppose that $c = [c_n] \in H^1(K_\mathfrak{p}, T)$ and $c^* = [c^*_n] \in H^1(K_\mathfrak{p}, T^*)$. Then*

$$(\kappa^*(c^*), c)_{\mathfrak{p},T} = -(c \cup c^*) \cdot w,$$

i.e.

$$\alpha_{\mathfrak{p},T}(\kappa^*(c^*), c) = [\alpha_{\mathfrak{p},E_{\pi^n}}(\kappa^*(c^*)_n, c_n)] = -(c \cup c^*).$$

Proof. Let

$$\cup : H^1(K_\mathfrak{p}, E_{\pi^n}) \times H^1(K_\mathfrak{p}, E_{\pi^{*n}}) \to \mathbf{Z}/p^n\mathbf{Z}$$

denote the cup product pairing 'at level n' afforded by (6.1). To prove the desired result, it suffices to show that, for each integer $n \geq 1$, we have

$$(r^*_n(c^*_n)(\pi^{-*n}w_n), c_n)_{\mathfrak{p},E_{\pi^n}} = -(c_n \cup c^*_n) \cdot w_n.$$

In order to do this, we first recall (see e.g. [17, Chapter XIV]) that the Hilbert pairing $(\ ,\ )_{\mathfrak{p},\mu_{p^n}}$ may be identified with a cup product pairing

$$\cup : H^1(\mathcal{N}_{n,\mathfrak{p}}, \mu_{p^n}) \times H^1(\mathcal{N}_{n,\mathfrak{p}}, \mu_{p^n}) \to \mathbf{Z}/p^n\mathbf{Z}$$

so that

$$(a, b)_{\mathfrak{p},\mu_{p^n}} = (a \cup b) \cdot \zeta_n.$$

We also note that the $\mathrm{Gal}(\overline{K}/\mathcal{N}_n)$-equivariant isomorphisms

$$E_{\pi^n} \xrightarrow{\sim} \mu_{p^n}; \quad w_n \mapsto e_n(w_n, \pi^{-n} w_n^*),$$
$$E_{\pi^{*n}} \xrightarrow{\sim} \mu_{p^n}; \quad w_n^* \mapsto e_n(\pi^{*-n} w_n, w_n^*)$$

induce isomorphisms of cohomology groups

$$\kappa_n : H^1(\mathcal{N}_{n,\mathfrak{p}}, E_{\pi^n}) \xrightarrow{\sim} H^1(\mathcal{N}_{n,\mathfrak{p}}, \mu_{p^n}); \quad y_n \mapsto r_n(y_n)(\pi^{-n} w_n^*);$$
$$\kappa_n^* : H^1(\mathcal{N}_{n,\mathfrak{p}}, E_{\pi^{*n}}) \xrightarrow{\sim} H^1(\mathcal{N}_{n,\mathfrak{p}}, \mu_{p^n}); \quad y_n^* \mapsto r_n^*(y_n^*)(\pi^{*-n} w_n),$$

and via functoriality of cup product pairings, we have

$$c_n \cup c_n^* = \kappa_n(c_n) \cup \kappa_n^*(c_n^*) = -(\kappa_n^*(c_n^*) \cup \kappa_n(c_n)).$$

Hence, from Lemma 5.6(b), we see that

$$\begin{aligned}
\alpha_{\mathfrak{p}, E_{\pi^n}}(r_n^*(c_n^*)(\pi^{-*n} w_n), c_n) &= \alpha_{\mathfrak{p}, \mu_{p^n}}(r_n^*(c_n^*)(\pi^{-*n} w_n), r_n(c_n)(\pi^{-n} w_n^*)) \\
&= \kappa_n^*(c_n) \cup \kappa_n(c_n) \\
&= -(\kappa_n(c_n) \cup \kappa_n^*(c_n^*)) \\
&= -(c_n \cup c_n^*),
\end{aligned}$$

as required. □

We can now state the main result of this section.

Proposition 6.2. *Suppose that* $c^* \in H^1(K_{\mathfrak{p}}, T^*)$ *with* $\kappa^*(c^*) \in U_{\infty,\mathfrak{p}} \subseteq \varprojlim H^1(\mathcal{K}_{n,\mathfrak{p}}, \mathbf{Z}_p(1))$. *Then*

$$\exp_{\mathfrak{p}}^*(c^*) = \left(\frac{\psi^*(\mathfrak{p})}{p} - 1 \right) \delta_w(\kappa^*(c^*)).$$

Proof. It follows from the definition of the dual exponential map (see [16, Section 5]) that

$$x \cup c^* = \log_{E,\mathfrak{p}}(x) \exp_{\mathfrak{p}}^*(c^*)$$

for all $x \in H^1(K_{\mathfrak{p}}, T)$. (Here we have extended the logarithm map $\log_{E,\mathfrak{p}}$ from $E(K_{\mathfrak{p}}) \otimes \mathbf{Z}_p$ to $H^1(K_{\mathfrak{p}}, T)$ via linearity.)

From Wiles's explicit reciprocity law (see [5, Chapter IV, §2.5] for the version we require), we have

$$(\kappa^*(c^*), c)_{\mathfrak{p},T} = \left\{ \log_{E,\mathfrak{p}}(x) \cdot \left(1 - \frac{\psi^*(\mathfrak{p})}{p} \right) \delta_w(\kappa^*(c^*)) \right\} \cdot w.$$

The result now follows from Proposition 6.1. □

7. Proof of Theorem A

In this section we shall establish a number of properties of the cohomology classes $s_{\mathfrak{p}}$ and $s_{\mathfrak{p}^*}$ constructed in Section 2, and thereby prove Theorem A of the Introduction. We shall require the following result.

Proposition 7.1. *Let $\overline{\sigma_{\mathfrak{p}}(w^*)} \in U^*_{\infty,\mathfrak{p}}$ be any lift of $\sigma_{\mathfrak{p}}(w^*)$. We have*

$$\mathcal{L}_{\mathfrak{p}}(\psi) = \left(1 - \frac{1}{\psi(\mathfrak{p}^*)}\right) \cdot \left(1 - \frac{1}{\psi(\mathfrak{p})}\right) \cdot \Omega_{\mathfrak{p}} \cdot \delta_w(\theta_B(w)). \tag{7.1}$$

and

$$\mathcal{L}'_{\mathfrak{p}}(\psi^*) = -\mathbf{N}(\mathfrak{f})^{-1} \cdot \left(1 - \frac{\psi^*(\mathfrak{p})}{p}\right) \cdot \lim_{n\to\infty} \left\{ \sum_{\tau \in \mathrm{Gal}(\mathcal{K}^*_n/K)} \psi^{*-1}(\tau) \log_{\mathfrak{p}}(\overline{\sigma_{\mathfrak{p}}(w^*)}^{\tau}_{n}) \right\}. \tag{7.2}$$

Proof. This follows directly from [12, Theorem 7.2], taking into account the fact that, by definition, we have $\sigma_{\mathfrak{p}}(w^*)^{\vartheta^*} = \theta_B(w^*)$ in $\overline{\mathcal{E}}^*_{\infty}/\mathcal{I}^*\overline{\mathcal{E}}^*_{\infty}$. □

Theorem 7.2. *We have*

$$\begin{aligned}\exp^*_{\mathfrak{p}}(s_{\mathfrak{p}^*}) &= \left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \cdot \delta_w(\theta_B(w)) \\ &= \left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \cdot \left(1 - \frac{1}{\psi(\mathfrak{p}^*)}\right)^{-1} \cdot \left(1 - \frac{1}{\psi(\mathfrak{p})}\right)^{-1} \cdot \frac{\mathcal{L}_{\mathfrak{p}}(\psi)}{\Omega_{\mathfrak{p}}}.\end{aligned}$$

Hence, if $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$, then $s_{\mathfrak{p}^*}$ is of infinite order.

Proof. This follows from Propositions 5.8, 6.2 and 7.1. □

We can now prove the first part of Theorem A.

Theorem 7.3. *Suppose that $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$.*

(a) We have $\mathcal{L}'_{\mathfrak{p}}(\psi^*) \neq 0$.

(b) The height pairing $[\,,\,]_{K,\mathfrak{p}}$ is non-degenerate, and we have

$$[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*} = (p-1) \cdot \mathbf{N}(\mathfrak{f})^{-1} \cdot \left(1 - \frac{1}{\psi(\mathfrak{p}^*)}\right)^{-1} \cdot \left(1 - \frac{1}{\psi(\mathfrak{p})}\right)^{-1} \cdot \mathcal{L}_{\mathfrak{p}}(\psi) \cdot \mathcal{L}'_{\mathfrak{p}}(\psi^*).$$

Proof. (a) Suppose that $\mathcal{L}'_{\mathfrak{p}}(\psi^*) = 0$, and set $t = \mathrm{ord}_{s=1} \mathcal{L}^*_{\mathfrak{p}}(s)$, so $t \geq 2$. Then it follows from [12, Theorem 7.2(i)] that

$$\overline{\mathcal{C}}^*_{\infty} \subseteq \mathcal{I}^{*t}(U^*_{\infty,\mathfrak{p}} \otimes \mathbf{Q}).$$

Hence we have that $\sigma_{\mathfrak{p}^*}(w^*) \in \mathcal{I}^{*(t-1)}(U^*_{\infty,\mathfrak{p}} \otimes \mathbf{Q}) \cdot \overline{\mathcal{E}}^*_{\infty}$, and this in turn implies that $s_{\mathfrak{p}^*} = 0$. This contradicts Theorem 7.2 because by hypothesis $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$. It therefore follows that $\mathcal{L}'_{\mathfrak{p}}(\psi^*) \neq 0$, as claimed.

(b) The equality follows directly from Theorem 5.9(b), Proposition 7.1, and Theorem 7.2. As $[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*} \neq 0$, we deduce that $[\ ,\ ]_{K,\mathfrak{p}^*}$ is non-degenerate because $\check{\Sigma}_{\mathfrak{p}}(K,T)$ and $\check{\Sigma}_{\mathfrak{p}^*}(K,T^*)$ are both rank one $\mathbf{Z}_p$-modules. □

The second part of Theorem A is a direct consequence of the following result.

Theorem 7.4. *(cf.* [12, Theorem 10.1]*)*

Suppose that $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$, and that $y \in \check{\Sigma}_{\mathfrak{p}}(K,T)$ and $y^* \in \check{\Sigma}_{\mathfrak{p}^*}(K,T^*)$ are both non-zero. Then

$$\frac{\exp^*_{\mathfrak{p}}(y^*) \cdot \exp^*_{\mathfrak{p}^*}(y)}{[y,y^*]_{K,\mathfrak{p}^*}} =$$

$$(p-1)^{-1} \cdot \mathbf{N}(\mathfrak{f}) \cdot \left(\frac{\psi^*(\mathfrak{p})}{p} - 1\right) \cdot \left(\frac{\psi(\mathfrak{p}^*)}{p} - 1\right) \cdot \frac{\mathcal{L}_{\mathfrak{p}^*}(\psi^*)}{(1-\psi^{*-1}(\mathfrak{p})) \cdot (1-\psi^{-1}(\mathfrak{p}^*)) \cdot \Omega_{\mathfrak{p}^*} \cdot \Omega_{\mathfrak{p}} \cdot \mathcal{L}'_{\mathfrak{p}}(\psi^*)}.$$

Proof. Since $\mathcal{L}_{\mathfrak{p}}(\psi) \neq 0$, it follows that

$$\mathrm{Hom}(\check{\Sigma}_{\mathfrak{p}}(K,T) \otimes_{\mathbf{Z}_p} \check{\Sigma}_{\mathfrak{p}^*}(K,T^*), \mathbf{Z}_p)$$

is a free $\mathbf{Z}_p$-module of rank one, and that both $\exp^*_{\mathfrak{p}^*}(\,) \cdot \exp^*_{\mathfrak{p}}(\,)$ and $[\ ,\ ]_{K,\mathfrak{p}^*}$ are non-zero elements of this module. Hence, since $s_{\mathfrak{p}}$ and $s_{\mathfrak{p}^*}$ are of infinite order, we have that

$$\frac{\exp^*_{\mathfrak{p}^*}(y) \cdot \exp^*_{\mathfrak{p}}(y^*)}{[y,y^*]_{K,\mathfrak{p}^*}} = \frac{\exp^*_{\mathfrak{p}^*}(s_{\mathfrak{p}}) \cdot \exp^*_{\mathfrak{p}}(s_{\mathfrak{p}^*})}{[s_{\mathfrak{p}}, s_{\mathfrak{p}^*}]_{K,\mathfrak{p}^*}}.$$

The desired result now follows from Theorem 7.2 and 7.3. □

This completes the proof of Theorem A.

References

[1] A. Agboola, *On Rubin's variant of the p-adic Birch and Swinnerton-Dyer conjecture*, Comp. Math. **143** (2007), 1374–1398.

[2] A. Agboola, *On certain special values of the Katz two-variable p-adic L-function*, in preparation.

[3] D. Bernardi, C. Goldstein, N. Stephens, *Notes p-adiques sur les courbes elliptiques*, Crelle **351** (1985), 129–170.

[4] A. Brumer, *On the units of algebraic number fields*, Mathematika **14** (1967), 121–124.

[5] E. de Shalit, *Iwasawa theory of elliptic curves with complex multiplication*, Academic Press (1987).

[6] R. Greenberg, *Trivial zeros of p-adic L-functions*, In: p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), 149–174, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[7] B. Mazur, J. Tate, J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), 1–48.

[8] B. Perrin-Riou, *Déscent infinie et hauteurs p-adiques sur les courbes elliptiques à multiplication complexe*, Invent. Math. **70** (1983), 369–398.

[9] B. Perrin-Riou, *Théorie d'Iwasawa et hauteurs p-adiques*, Invent. Math. **109** (1992), 137–185.

[10] B. Perrin-Riou, *Arithmétique des courbes elliptiques et théorie d'Iwasawa*, Memoire de la Société Mathématique de France, **17** (1984).

[11] K. Rubin, *Tate-Shafarevich groups and L-functions of elliptic curves with complex multiplication*, Invent. Math. **89** (1987), 527–560.

[12] K. Rubin, *p-adic L-functions and rational points on elliptic curves with complex multiplication*, Invent. Math. **107** (1992), 323–350.

[13] K. Rubin, *p-adic variants of the Birch and Swinnerton-Dyer conjecture for elliptic curves with complex multiplication.* In: p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), 71–80, Contemp. Math., 165, Amer. Math. Soc., Providence, RI, 1994.

[14] K. Rubin, *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, Invent. Math. **109** (1992), 25–68.

[15] K. Rubin, *Euler Systems*, Princeton University Press (2000).

[16] K. Rubin, *Euler systems and modular elliptic curves.* In: Galois representations in arithmetic geometry, A.J. Scholl, R. L. Taylor (Eds.), CUP 1998.

[17] J.-P. Serre, *Local Fields*, Springer Verlag (1979).

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