MATH 137B Midterm

Friday, May 6, 2011

Solutions

1. Let G and G' be 1-isomorphic graphs on n and n' vertices and with c and c' components. Prove that $n - c = n' - c'$.

   Solution: By successive 1-split operations, the graph G can be reduced to the graph H, where each component of H is a block of G, and vice versa. By the definition of 1-isomorphic, the graph G' can similarly be reduced to the graph H.

   After performing a 1-split operation on a graph G, the resulting graph gains one new component for each new vertex, and vice versa. Therefore, *difference* between the number of vertices and the number of components remains constant.

   Hence, if H has n'' vertices and c'' components then

$$n - c = n'' - c''$$

   and

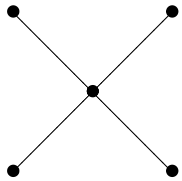
$$n' - c' = n'' - c''.$$

   Thus,

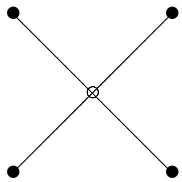
$$n - c = n' - c'.$$

2. Find a graph on five vertices that is isomorphic to its block-cutpoint graph.

   Solution: Consider the star graph $K_{1,4}$

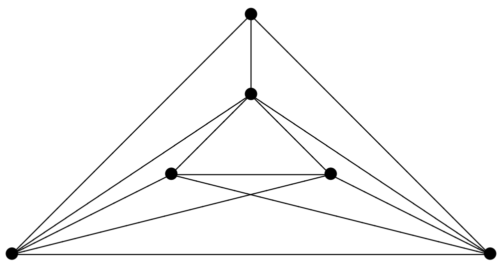


   The center vertex is the only cutpoint and each edge is a block of $K_{1,4}$. Hence, $\mathrm{BC}(K_{1,4})$ is the graph



   which is clearly isomorphic to $K_{1,4}$.

3. Determine whether or not the graph given below is planar.



Justify your answer in *two different* ways.

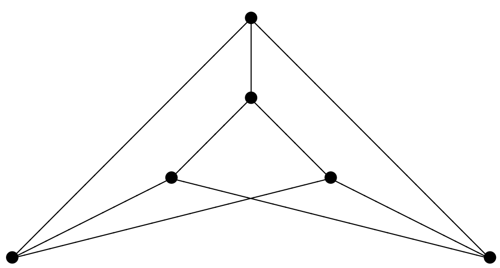
Solution:

(i) By Corollary 7.15 of the text, if G is a simple planar graph on $n \geq 3$ vertices with m edges then

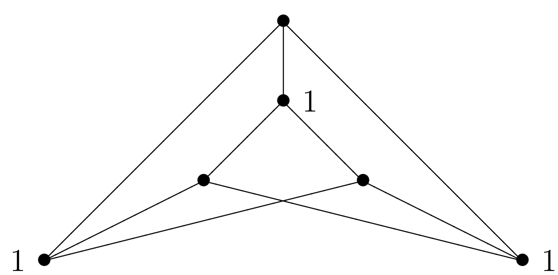
$$m \leq 3n - 6.$$

As the given graph has 6 vertices and 13 edges and $13 > 3 \times 6 - 6 = 12$, it is not planar.

(ii) Removing four select edges gives the following minor

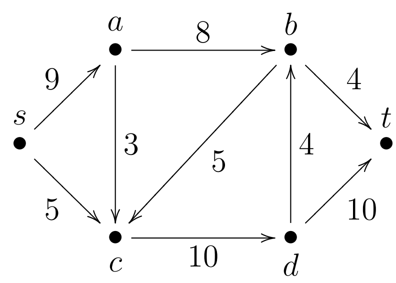


To see that this new graph is isomorphic to $K_{3,3}$, we label one bipartition set.



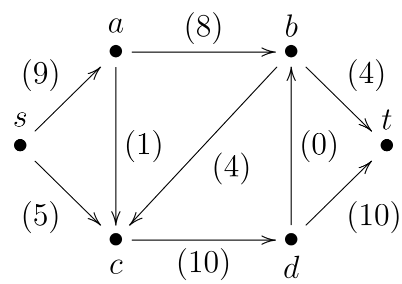
Thus, by Wagner's Theorem, the given graph is not planar.

4. Consider the network given below.



Find a maximum flow. Justify your answer.

Solution: Consider the indicated function on the set of edges of the network graph.



This function is indeed a flow, as for each edge the function value is less than or equal to capacity and for each vertex (except for the source and sink) the Kirchhoff conditions holds.

The flow is maximum: Since the flow is equal to capacity for all edges adjacent to the source, there can be no augmenting path.