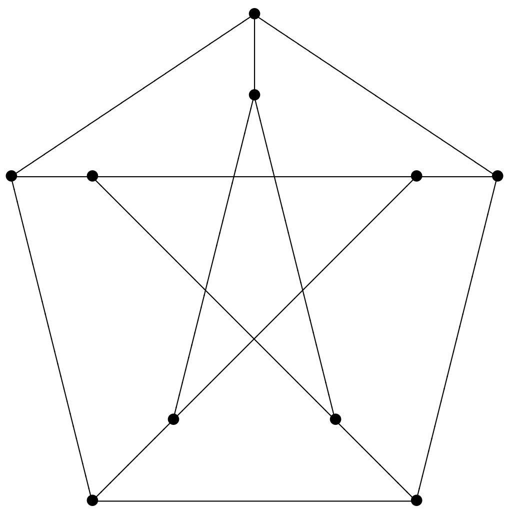
HOMEWORK 4

SOLUTIONS

(1) Determine the chromatic number of the Petersen graph.



Solution: The Petersen graph contains a cycle of odd length as a subgraph. Hence,

$$3 \leq \chi(C_5) \leq \chi(P).$$

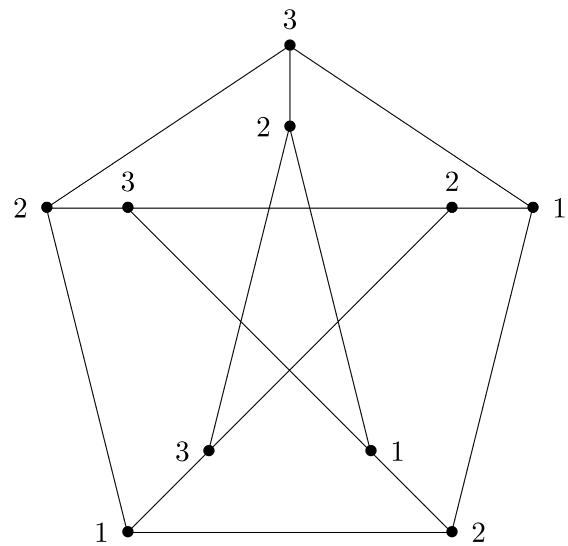
As the Petersen graph is neither a complete graph nor itself a cycle of odd length, we can invoke Brooks's Theorem:

$$\chi(P) \leq \Delta(P) = 3.$$

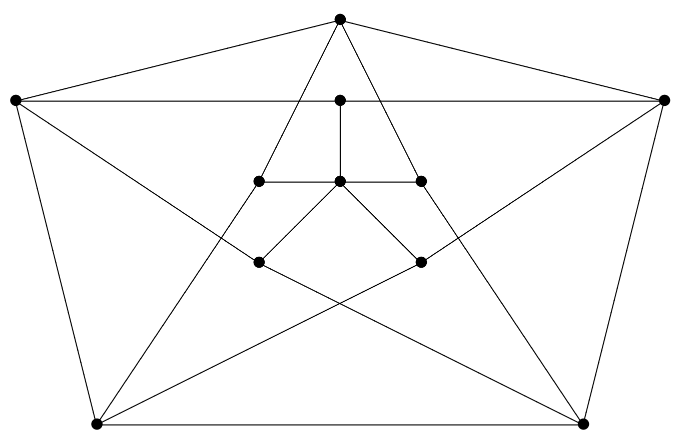
Thus,

$$\chi(P) = 3.$$

We demonstrate a proper 3-vertex coloring below.



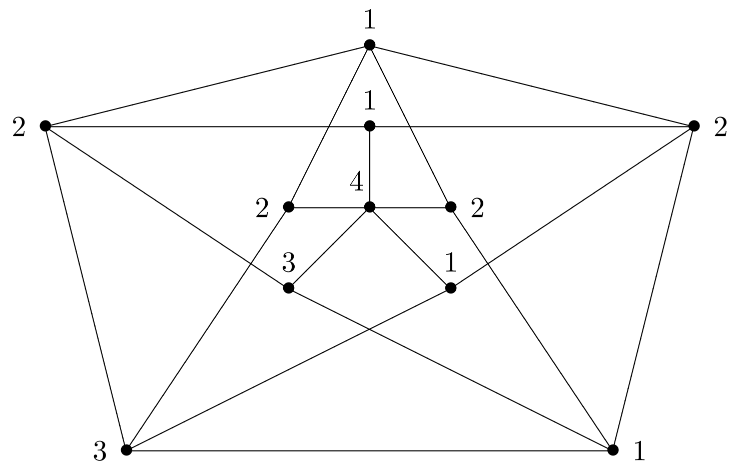
(2) Determine the chromatic number of the Grötzsch graph.



Solution: As in the solution to Problem 1, Brooks' theorem and the fact that the Grötzsch graph has C_5 as a subgraph gives

$$3 \leq \chi(G) \leq 5.$$

Below is a proper 4-vertex coloring for the Grötzsch graph.



Hence,

$$3 \leq \chi(G) \leq 4.$$

That is, $\chi(G)$ is either 3 or 4.

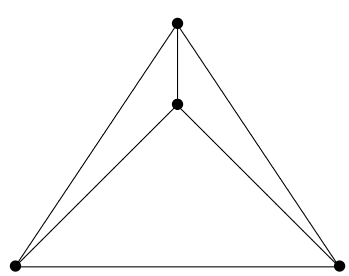
Now suppose there existed a proper 3-vertex coloring for the Grötzsch graph. WLOG, we can assume that the center vertex has coloring 3. This forces the five neighbors of the center vertex to have coloring 1 or 2. This, in turn, creates a conflict with the coloring of the 5-cycle that bounds the Grötzsch graph, which as an odd cycle, requires at least 3 colors.

Therefore, $\chi(G) = 4$.

(3) Draw a self-dual plane graph on four vertices.

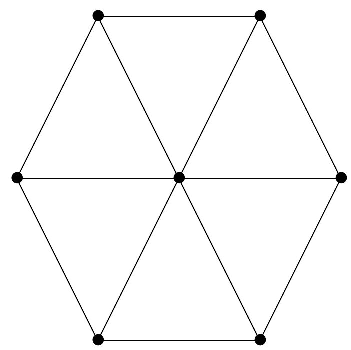
Solution: For any graph isomorphic to its plane dual, the number of vertices must equal the number of faces. So we are looking for a graph with four vertices and four faces. Therefore, the complete graph K_4 is a reasonable candidate.

Remember, when dealing with plane dual the embedding (how a graph is drawn) matters. We consider the standard plane embedding of K_4:



The plane dual of this graph will have four vertices and six edges, as does the original graph. Note that every face of K_4 (including the infinite face) is bounded by 3 edges. This tells that the degree of each vertex in K_4^* is 3. It is also clear that K_4^* is simple. A simple graph on four vertices where every vertex has degree 3 is isomorphic to K_4.

(4) Draw a self-dual plane graph on seven vertices.
Solution: Using similar considerations as above, we obtain the following self-dual plane graph on seven vertices.



(5) For a simple connected graph G, with at least two vertices, prove that $\chi(G) = 2$ if and only if G is bipartite.
Solution: Assume G is bipartite with bipartition $V(G) = X \cup Y$. Assign color 1 to all the vertices in X and color 2 to all the vertices in Y. Since any edge in G has exactly one endvertex in each set, G has a proper 2-vertex coloring. We have to check that $\chi(G) \neq 1$. But this is only possible if G has no edges.
Now assume $\chi(G) = 2$. Let $X \subset V(G)$ be the set of vertices colored 1 and $Y \subset V(G)$ the set of vertices colored 2. Clearly, $X \cup Y = V(G)$ and $X \cap Y = \emptyset$. Any edge in G must have one vertex in X and the other in Y since G has a proper 2-vertex coloring. Thus X and Y form a bipartition for G.

(6) For a simple connected graph G, with at least two vertices, prove that $\chi(G) \leq k$ if and only if G is k-partite.
Solution: Assume G is k-partite. Assign the colors $1, \cdots, k$ to the partition sets. As above, this defines a proper k-vertex coloring for G. Note that this does not imply that $\chi(G) = k$, only that $\chi(G) \leq k$. But that is all we are asked to prove.
Now assume that $\chi(G) \leq k$. Clearly, by sorting vertices by color, we get a $\chi(G)$-partition. By subdividing partition sets (if we have enough vertices) or allowing empty sets (if we don't) we can increase the number of partition sets to any number, including k.

(7) For a simple connected graph G, with n vertices, prove that $\chi(G) = n$ if and only if $G = K_n$.

Solution: Clearly, $G = K_n \Rightarrow \chi(G) = n$. So we need only prove the other direction. Here, we prove the contrapositive.

Let G be a graph on n vertices such that $G \neq K_n$. Then, there exists $u \in V(G)$ such that $d_G(u) < n - 1$.

Now $G - u$ is a simple graph on $n - 1$ vertices, therefore

$$\chi(G-u) \leq \Delta(G-u) + 1 = (n-2) + 1 = n - 1.$$

We can then extend the at most $n - 1$ coloring of $G - u$ to G without adding any additional colors.

(8) Let G be a simple graph on n vertices and $\overline{G}$ its complement. Show that

$$\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}.$$

Solution: First we show that

$$n = \chi(K_n) \leq \chi(G) \cdot \chi(\overline{G}).$$

That is, we can color the complete graph with $\chi(G) \cdot \chi(\overline{G})$ colors.

Color the complete graph with the $\chi(G)$ colors. Clearly, this is not yet a proper coloring. The vertices which were colored 1 in G are now all pairwise adjacent by the addition of the edges from $\overline{G}$. Recolor all those vertices with $\chi(\overline{G})$ different colors. This is enough since $\overline{G}$ has a $\chi(\overline{G})$-vertex coloring.

Move on to the vertices colored 2 by G. Recolor these vertices by a *new* set of $\chi(\overline{G})$ colors. And so on.

This process is, of course, overkill. But in the end, no adjacent vertices will have the same color. Hence we have a proper $\chi(G) \cdot \chi(\overline{G})$-vertex coloring of K_n.

Now for any two positive numbers x and y,

$$\begin{aligned}
(x-y)^2 \geq 0 \quad &\Rightarrow \quad x^2 + y^2 - 2xy \geq 0 \\
&\Rightarrow \quad x^2 + y^2 + 2xy \geq 4xy \\
&= \quad (x+y)^2 \geq 4xy \\
&= \quad x + y \geq 2\sqrt{xy}
\end{aligned}$$

So we have,

$$\chi(G) + \chi(\overline{G}) \geq 2\sqrt{\chi(G) \cdot \chi(\overline{G})} = 2\sqrt{n}.$$