HOMEWORK 1

SOLUTIONS

(1) Determine all $m, n \in \mathbb{N}$ such that the complete bipartite graph $K_{m,n}$ is Hamiltonian.
Solution:

Claim 1. *The complete bipartite graph $K_{n,n}$ is Hamiltonian, for all $n \geq 2$.*

Proof. $K_{n,n}$ is a simple graph on $2n$ vertices. So for $n \geq 2$, we have that $K_{n,n}$ has at least 3 vertices. Under this restriction, a sufficient condition for Hamiltonicity is that the degree of every vertex is greater than or equal to half the number of vertices. As the degree of each vertex in $K_{n,n}$ is n $(= 2n/2)$, we have our desired result. □

We note here that for $n = 1$ or 2, $K_{n,n}$ is a tree, and is therefore not Hamiltonian.

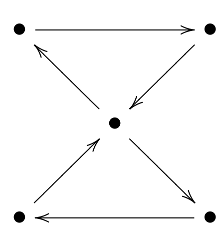
Claim 2. *The complete bipartite graph $K_{m,n}$ is not Hamiltonian when $m \neq n$.*

Proof. WLOG we assume that $n < m$. Let $V' \subseteq V(K_{m,n})$ be the bipartition set of order n. A necessary condition for the Hamiltonicity of a simple graph G is that for each nonempty $S \subseteq V(G)$, the number of components of $G - S$ is less than or equal to the order of S.

For $S = V'$, $G - S$ is the null graph on m vertices. Hence, the number of components of $G - S$ is m, which is *greater* than $n = |S|$. Thus, $K_{m,n}$ is not Hamiltonian for $m \neq n$. □

(2) Give an example of a strongly connected digraph whose underlying graph is not Hamiltonian.
Solution: One way to think of strongly connected is that the graph is in some way a composition of cycles. In that sense, one can reach any point from any other. To frustrate Hamiltonicity, there must not be one "big" cycle, that is, a cycle that includes all vertices.

This line of thought leads us to the idea of a "bottleneck" and the example below.



(3) Let $\vec{G}$ be a digraph on $n \geq 2$ vertices. Let

$$Y = A(\vec{G}) + A(\vec{G})^2 + A(\vec{G})^3 + \cdots + A(\vec{G})^{n-1}.$$

Prove that $\vec{G}$ is strongly connected if, and only if, all entries of Y that do not lie on the main diagonal are nonzero.

Solution: We begin with an observation for digraphs that is analogous to what was described in the undirected case last term.

If $\vec{G}$ is a digraph on n vertices and a_{ij} and $a_{ij}^{(2)}$ are the ij-entries of $A(\vec{G})$ and $A(\vec{G})^2$ respectively, then by the rules of matrix multiplication

$$a_{ij}^{(2)} = a_{i1}a_{1j} + a_{i2}a_{2j} + \cdots + a_{in}a_{nj}.$$

Thus, $a_{ij}^{(2)}$ counts the number of directed walks of length two from the vertex of $\vec{G}$ labeled u_i to the vertex labeled u_j. We can make a similar statement for $a_{ij}^{(k)}$, $1 \leq k \leq n-1$. Thus, the ij-entry of Y,

$$y_{ij} = a_{ij} + a_{ij}^{(2)} + \cdots + a_{ij}^{(n-1)},$$

counts the number of directed walks in $\vec{G}$ from u_i to u_j of length less than n.

Now suppose that for each distinct pair i, j, $y_{ij} \neq 0$. Then, there is at least 1 directed walk between u_i and u_j, for all i, j, $i \neq j$. Hence, $\vec{G}$ is strongly connected.

On the other hand, assume $\vec{G}$ is strongly connected. Then, there exists a directed walk from any vertex of $\vec{G}$ to any other. This walk is of positive length, if the vertices are distinct. Another result for digraphs analogous to an established result for undirected graphs is that the existence of a directed walk of positive length implies the existence of a directed path. Since $\vec{G}$ has n vertices, such a path has length less than n. At least one path of length less than n for every pair of distinct vertices implies that $y_{ij} \neq 0$ for all $i \neq j$.

(4) Show that there exists a vertex labeling of $\vec{G}$ such that $A(\vec{G})$ is a strictly lower triangular matrix if, and only if, $\vec{G}$ is an acyclic digraph.

Solution: Assume there exists a vertex labeling of $\vec{G}$ such that $A(\vec{G})$ is a strictly lower triangular matrix. This means that, a_{ij}, the ij-entry of $A(\vec{G})$, is nonzero if, and only if, $i < j$. Thus, there is a directed edge between u_i and u_j only when $i < j$. Hence, a directed walk in $\vec{G}$ can never repeat a vertex. Therefore, $\vec{G}$ contains no directed cycles.

Now if $\vec{G}$ is an acyclic digraph, by a result proven in class, there exists a partial order $\preceq$ on $V(\vec{G})$ given by

$$u \preceq v \Leftrightarrow \text{there is a directed path from } v \text{ to } u \text{ in } \vec{G}.$$

A vertex $v \in V(\vec{G})$ is maximal with respect to the partial order $\preceq$ if $\nexists\, u \in V(\vec{G})$ such that $v \preceq u$. As $V(\vec{G})$ is assumed to be finite, there exists at least one maximal element. Give this vertex the label 1.

If there is another maximal vertex, give it the label 2. Continue until all maximal elements all labeled. Next, label the vertices that are adjacent to the maximal vertices. Following this labeling, a directed edge from u_i to u_j will only exist if $i < j$. Hence, $A(\vec{G})$ will be strictly lower triangular.

(5) Let G be a graph with n vertices, where $n \geq 2$. Prove that G has at least two vertices which are not cut vertices.

Solution: Proof by contradiction. If G does not have at least two vertices which are not cut vertices, then G has 0 or 1 vertices which are not cut vertices. In other words, G has at most one vertex that is not a cut vertex.

Now, suppose G is connected. Let u and v be vertices in G such that the distance between them is maximal. That is, the shortest path from u to v is longer than the shortest path between any pair of distinct vertices in G. Since G has at least 2 vertices, $u \neq v$. By assumption, one of these vertices, say u, is a cut vertex.

Hence $G - u$ is disconnected. Let $w \in V(G - u)$ be such that w and v lie in different components of $G - u$. Therefore, any path from w to v contains u. Thus, the shortest path from w to v in G contains the shortest path from u to v. Contradiction.

If G is disconnected, do the same proof on any connected component with two or more vertices. If all the components of G contain only one vertex, then G has no cut vertices.

(6) Let v be a cut vertex of a simple connected graph G. Prove that v is not a cut vertex of its complement $\overline{G}$.

Solution: Consider the connected components of $G - v$. By definition of complement, any pair of vertices lying in distinct components are adjacent in $\overline{G}$. Hence, any pair of vertices in $\overline{G} - v$ are connected by a path of length at most two. Thus, v is not a cut vertex of $\overline{G}$.

(7) Let G be a simple connected graph with at least two vertices. Prove that

$$\kappa(G) \leq \frac{2m}{n},$$

where m is the number of edges and n is the number of vertices.

Solution: We showed in class that the connectivity of a graph G is less than or equal to the minimum degree of the vertices. That is,

$$\kappa(G) \leq \delta(G).$$

By the Hand-Shaking Theorem

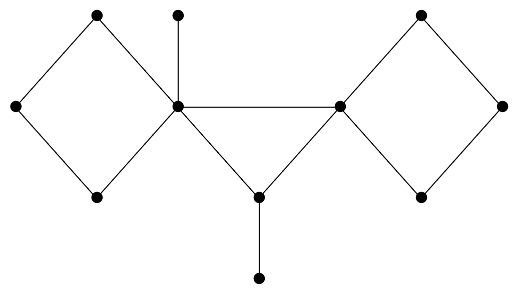
$$n\delta(G) = \sum_{u \in V(G)} \delta(G) \leq \sum_{u \in V(G)} d_G(u) = 2|E(G)| = 2m.$$

Therefore,

$$\delta(G) \leq \frac{2m}{n}.$$

(8) Draw the block cut-point graph for the graph G.

G :



Solution: