MATH 137A Midterm

Friday, February 11, 2011

Solutions

1. Determine whether K_4 contains the following (give an example or a proof of non-existence).

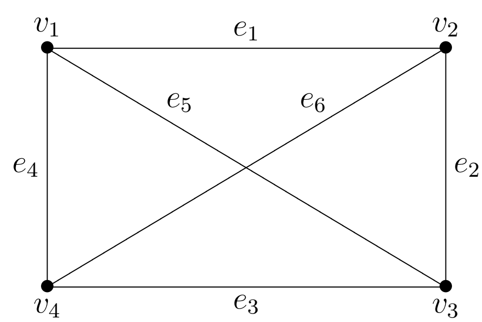
   (a) A walk that is not a trail.

   (b) A trail that is not closed and is not a path.

   (c) A closed trail that is not a cycle.

Solution:

A labeled complete graph on four vertices is given below.



(a) A trail is a walk with all of its edges distinct. Therefore

$$w = (v_1, e_1, v_2, e_6, v_4, e_3, v_3, e_2, v_2, e_1, v_1)$$

is one example of a walk that is not a trail.

(b) The walk

$$w = (v_1, e_5, v_3, e_2, v_2, e_6, v_4, e_3, v_3)$$

is a trail, since no edges are repeated. It is not closed, since the initial and final vertices are not the same. It is not a path, since the vertex v_3 is repeated.

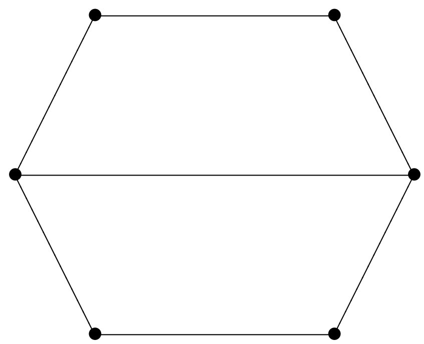
(c) There is no closed trail in K_4 that is not a cycle. Such a walk would have to repeat no edge (so as to be a trail) and at the same time repeat a vertex, say u, that is not the initial/final vertex (so as to not be a cycle).

Similar to the case of the Bridges of Königsberg problem, the degree of u would have to be at least four. This is because u would have to be "entered" then "exited" then "reentered" then "reexited", all on different edges.

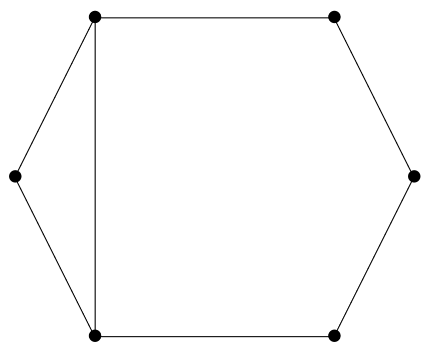
Since the maximum degree of a vertex in K_4 is 3, no such walk exists.

2. Find three nonisomorphic simple graphs on six vertices where two vertices have degree 3 and four vertices have degree 2.

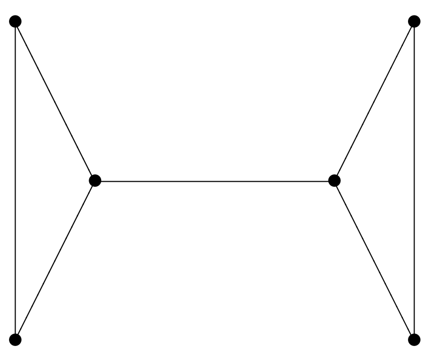
   Solution: Adding an edge to C_6, the cycle graph on 6 vertices, gives one example.

   Adding the edge in a different way gives another example.

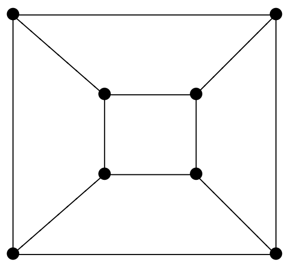
   

   To get a third example, we connect two C_3 graphs together.

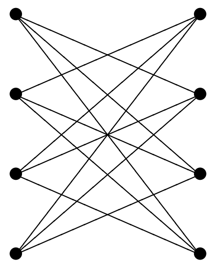
   

   Note that no two graphs are isomorphic as the first graph has no 3-clique, the second graph has one 3-clique and the third graph has two 3-cliques.

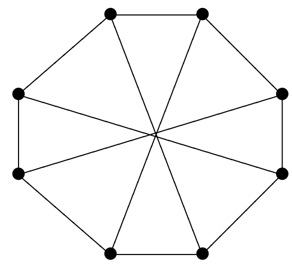
3. Determine which pairs of graphs below are isomorphic. Justify your answer.



A

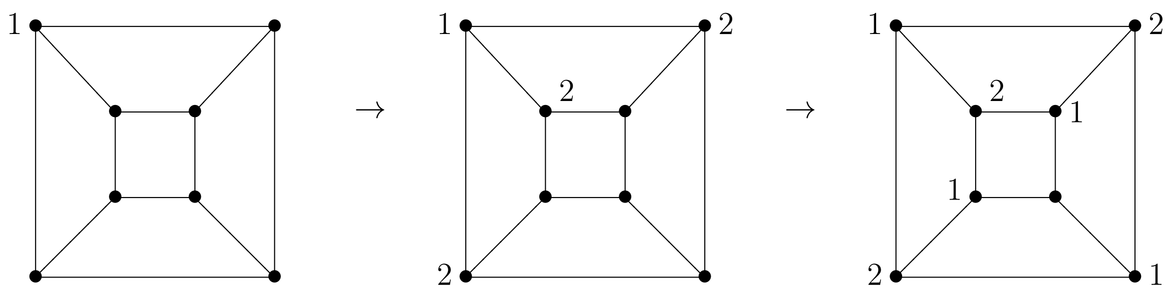


B



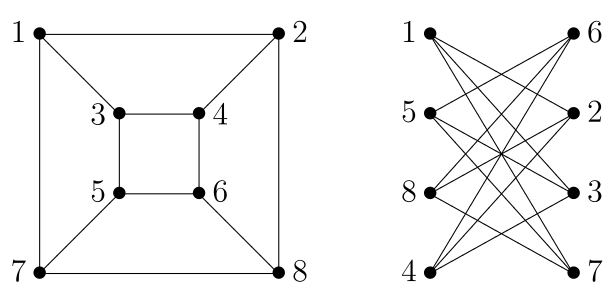
C

Solution: Graph B is clearly bipartite. So is graph A. To see this, start with any vertex you like and label it 1. Then label all adjacent vertices 2. Continue to alternately label adjacent vertices until each vertex of A is assigned a 1 or a 2. As this can be done without contradiction, we establish a bipartition for $V(A)$. See below.



This process won't work for graph C (try it!). Hence, graph C is not bipartite and is therefore not isomorphic to graphs A or B.

Since A and B are simple graphs, to establish an isomorphism between them we need only to demonstrate a correspondence between the vertex sets that respects the edge sets. We do so below. Check that two vertices are adjacent on the left if and only if the corresponding vertices on the right are also adjacent.



4. A graph G is called **unicyclic** if it is connected and contains precisely one cycle. Prove that if a connected graph G with n vertices and e edges is unicyclic then $n = e$.

   Solution: Let G be unicyclic and let f be an edge in the one cycle of G. Then the graph $H = G - f$ has $e - 1$ edges.

   On the other hand, H is a connected graph: any u, v-path that contained e in G can be replaced in H using the other edges of the cycle. Furthermore, since G had only one cycle, H has none. Thus, H is a tree on the same n vertices as G, and so, H has $n - 1$ edges.

   Hence, $e - 1 = n - 1$. Therefore, $e = n$.

5. Let T be a tree on n vertices, $n \geq 2$, where each vertex is of degree three or less. Let x, y and z denote the number of vertices in T of degree 1, 2 and 3, respectively.

   (a) Show that $x + y + z = n$.

   (b) Show that $x + 2y + 3z = 2n - 2$.

   (c) Use parts (a) and (b) to show that $z \leq \frac{n}{2} - 1$.

   Note that this problem gives a simpler proof of a result from class.

Solution:

(a) Since T is a tree, it is connected. Hence, there are no vertices of degree zero. Thus, the number of vertices of degree one, x, plus the number of vertices of degree two, y, plus the number of vertices of degree three, z, equals the total number of vertices, which is n.

(b) By the Hand-Shaking Theorem, the sum of the degrees of the vertices in T equals twice the number of edges in T. In class we proved that a tree on n vertices has exactly $n - 1$ edges. Twice $n - 1$ is $2n - 2$. On the other hand, the sum of the degrees of the vertices is the number of vertices of degree one times one, plus the number of vertices of degree two times two, plus the number of vertices of degree three times three.

(c) We have simultaneous equations

$$\begin{aligned} x + 2y + 3z &= 2n - 2 \\ x + y + z &= n \end{aligned}$$

Rewrite the first equation as

$$(x + y + z) + y + 2z = 2n - 2$$

Then substitute using the second equation.

$$n + y + 2z = 2n - 2$$

Thus

$$\begin{aligned} & y + 2z = n - 2 \\ \Rightarrow\ & 2z \leq n - 2 \text{ (since } y \geq 0) \\ \Rightarrow\ & z \leq \frac{n}{2} - 1 \end{aligned}$$

6. The Full Binary Tree Theorem is stated as follows:

Theorem (Full Binary Tree)**.** *A regular binary tree has $n = 2k + 1$ vertices, where k of them are internal and $k + 1$ of them are leaves.*

We proved this theorem in class using the Principle of Mathematical Induction. This problem gives an alternate proof.

(a) Recall that a nontrivial regular binary tree has exactly one vertex of degree two and all the other vertices have degree one or three. Use a corollary to the Hand-Shaking Theorem to show that a regular binary tree must have an odd number of vertices. Conclude that if T is a regular binary tree on n vertices, we may write $n = 2k + 1$ for some $k \in \mathbb{N}$.

(b) If a tree has $2k + 1$ vertices, how many edges does it have?

(c) Let w be the number of leaves (i.e., vertices of degree one) in a regular binary tree. Use parts (a) and (b) and the Hand-Shaking Theorem to write an equation in w and k.

(d) Solve the equation you wrote in part (c) for w.

Solution:

(a) The corollary in question states that in any graph the number of vertices having odd degree is even. Since all but one of the vertices of a regular binary tree T have odd degree, $|V(T)|$ must be an odd number. Therefore, if n is the number of vertices of T, we may write $n = 2k + 1$ for some $k \in \mathbb{N}$.

(b) If a tree has $n = 2k+1$ vertices, then it must have $n-1 = 2k+1-1 = 2k$ edges.

(c) A (nontrivial) regular binary tree has one root with degree 2. If it also has w leaves, then remaining $2k+1-1-w = 2k-w$ vertices all have degree 3. By the the Hand-Shaking Theorem, we have

$$1 \times w + 2 \times 1 + 3 \times (2k - w) = 4k$$

or

$$w + 2 + 6k - 3w = 4k$$

(d) Part (c) and a little algebra shows that $w = k + 1$.