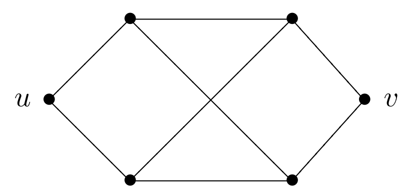
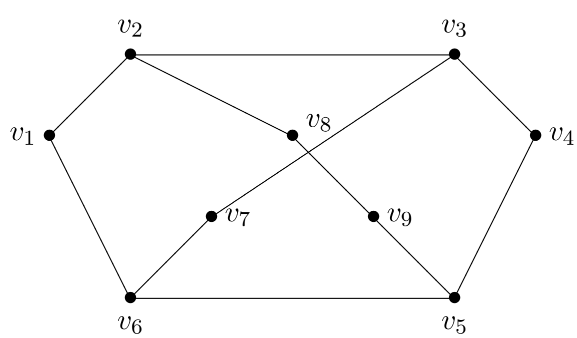
HOMEWORK 5

SOLUTIONS

(1) Compute the total resistance between u and v in the electrical network of resistors shown below, where one diagonal resistor is two ohms, the other diagonal resistor is three ohms, and the remaining resistors are one ohm.



Solution: First we subdivide the resistors to get a graphical representation where all the edges represent a resistance of one ohm. Then we label the vertices.



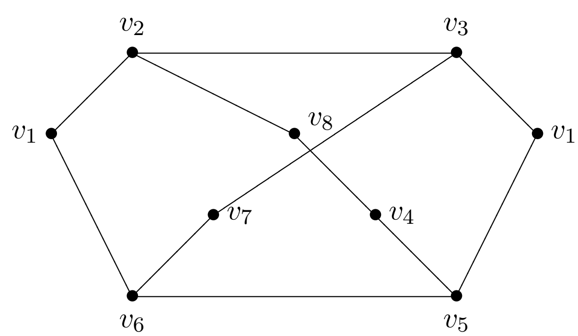
So we have

$$D(G) - A(G) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Hence

$$\tau(G) = \begin{vmatrix} 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 93$$

The graph G'', the result of identifying u and v in G, can be represented by giving the two vertices the same label.



Thus

$$D(G'') - A(G'') = \begin{pmatrix} 4 & -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & -1 & 3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

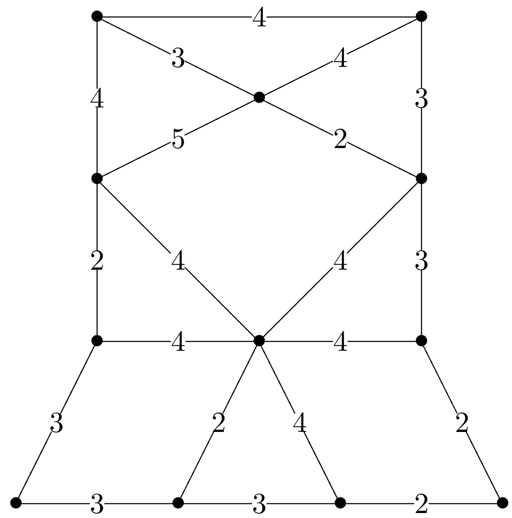
And we have

$$\tau(G'') = \begin{vmatrix} 4 & -1 & -1 & 0 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 & 2 \end{vmatrix} = 126$$

Therefore, the total resistance between u and v is

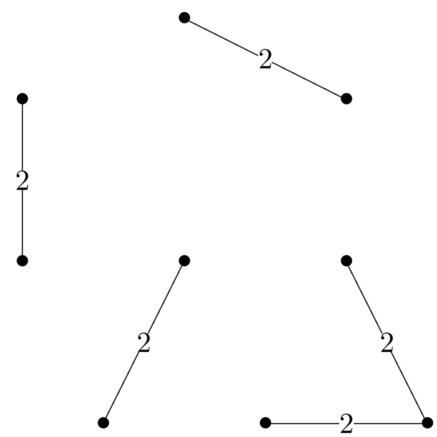
$$\frac{126}{93} = \frac{42}{31}.$$

(2) Find a minimal cost spanning tree using both Kruskal's and Prim's algorithms.

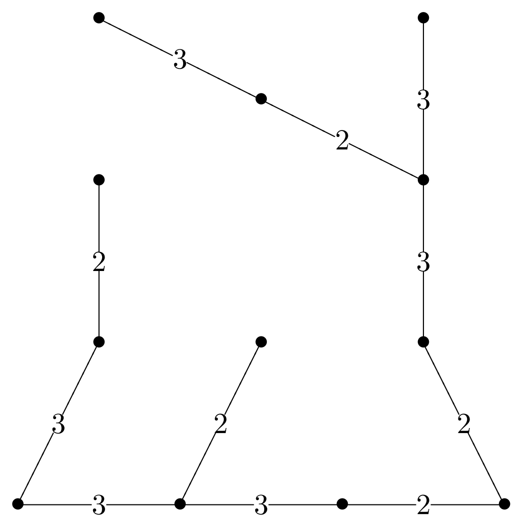


Solution: Kruskal's algorithm begins by selecting a minimum weighted edge. We have 5 choices for an edge of weight 2. Here, we pick the edge on the left hand side.

We continue to select edges with weight 2. As no cycles result by doing so, the next four steps of the Kruskal algorithm gives the following forest.

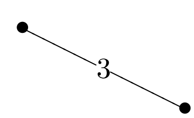


We then select edges with weight 3. Again, as no cycles are created, the result of the next six steps is as given below.

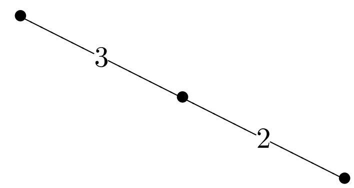


The process is complete since the addition of any other edge would create a cycle. Alternatively, we could note that the original graph had 12 vertices and we have already selected $11 = 12 - 1$ edges.

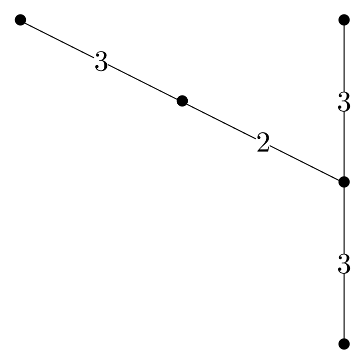
Prim's algorithm begins with any vertex. Here, we choose the upper-leftmost vertex and then the minimum weight edge incident to it.



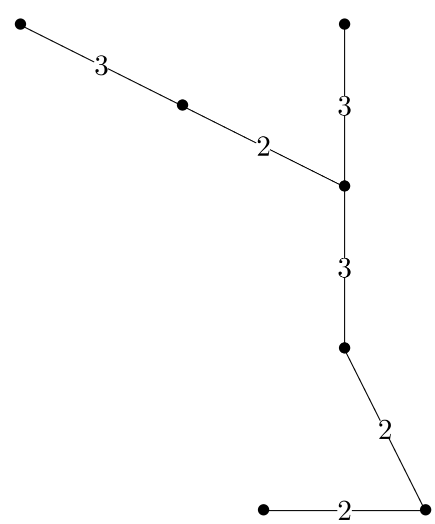
Of all the remaining edges that share exactly one endvertex with the original edge, only one has weight 2.



We now consider all the edges of the original graph that have exactly one endvertex from the three vertices above. In doing so, it is not hard to see that the next two steps of Prim's algorithm yields the following.



And the two steps after that give:



At this point it is clear that Prim's algorithm will continue on to give the same minimum cost spanning tree, of weight 28, achieved by Kruskal's algorithm.

(3) Prove, directly from the definition, that every subgraph of a bipartite graph is also bipartite.

Solution: A graph G is bipartite if there exists nonempty sets X and Y such that $V(G) = X \cup Y$, $X \cap Y = \emptyset$ and each edge in G has one endvertex in X and one endvertex in Y. Let G be a bipartite graph and let H be a subgraph of G.

Now, let $X' := V(H) \cap X$ and $Y' := V(H) \cap Y$. Then

$$X' \cup Y' = [V(H) \cap X] \cup [V(H) \cap Y] = V(H) \cap [X \cup Y] = V(H) \cap V(G) = V(H)$$

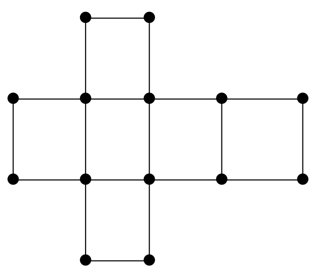
and

$$X' \cap Y' = [V(H) \cap X] \cap [V(H) \cap Y] = V(H) \cap [X \cap Y] = V(H) \cap \emptyset = \emptyset.$$

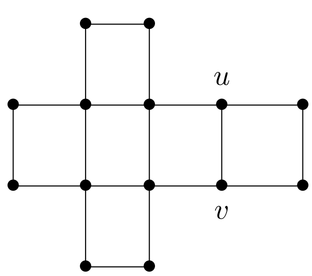
Furthermore, there can be no edge in H with both endpoints in X', since that would mean that there exists an edge in G with both endpoints in X. Similarly, there can be no edge in H with both endpoints in Y'.

If either X' or Y' should be empty, then H is in fact a null graph and, again, bipartite.

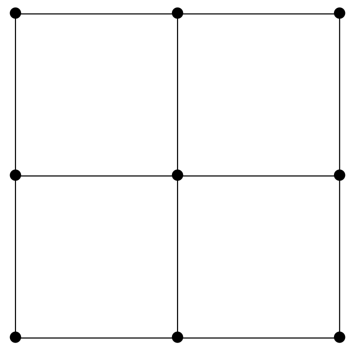
(4) Determine whether the given graph is Eulerian. If it is, find an Eulerian circuit. If it is not, prove it is not.



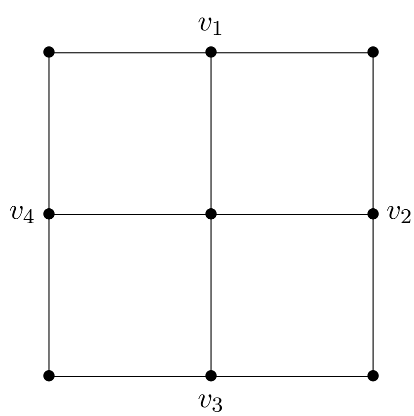
Solution: An Eulerian circuit is a closed trail that contains each edge of a graph. We saw in class that a connected graph is Eulerian if, and only if, each vertex has even degree. The vertices indicated below both have degree 3, which is odd. Hence, the graph cannot be Eulerian.



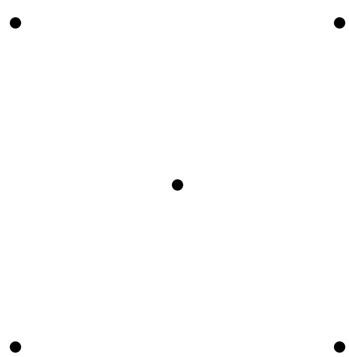
(5) Determine whether the given graph is Hamiltonian. If it is, find a Hamiltonian cycle. If it is not, prove it is not.



Solution: We saw in class that a necessary condition for Hamiltonicity is that the number of components of $G - S$ is less than $|S|$, for all $S \subseteq V(G)$. Let S be the set of vertices of G indicated below by v_1, v_2, v_3, and v_4.



Then, $G - S$



has 5 components. As $5 \not\leq 4$, G is not Hamiltonian.