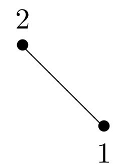
HOMEWORK 4

SOLUTIONS

(1) Draw the tree whose Prüfer code is $(1, 1, 1, 1, 6, 5)$.

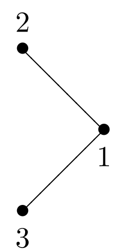
Solution: The given Prüfer code has six entries, therefore the corresponding tree will have $6 + 2 = 8$ entries.

The first number in the Prüfer code is 1 and the lowest number not included in the Prüfer code is 2, so we connect 1 to 2.



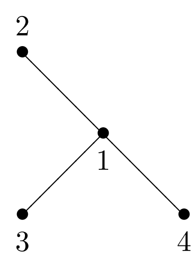
We then drop the leading 1 in the code and put 2 at the back of the code: $(1, 1, 1, 6, 5, 2)$.

The first number in the code is still 1 and the lowest number not included is now 3, so we connect 1 to 3.



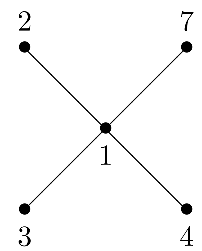
We, again, drop the leading 1 and then put 3 at the back of the code: $(1, 1, 6, 5, 2, 3)$.

The first number in the code is still 1 and the lowest number not included is now 4, so we connect 1 to 4.



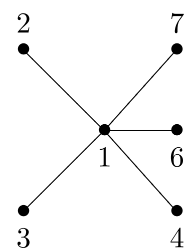
We drop the leading 1 and put 4 at the back of the code: $(1, 6, 5, 2, 3, 4)$.

The first number in the code is, yet again, 1 and the lowest number not included is now 7. So we connect 1 to 7.



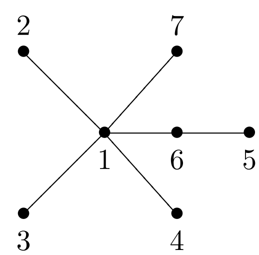
We drop the leading 1 an put 7 at the back of the code: $(6, 5, 2, 3, 4, 7)$.

The first number in the code is now 6 and the lowest number not included is 1, so we connect 6 to 1.



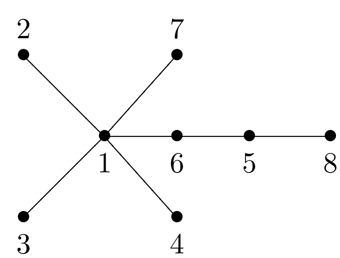
We drop the 6 from the code and put 1 at the back: $(5, 2, 3, 4, 7, 1)$.

Now the first number in the code is 5 and the lowest number not included is 6, so we connect 5 to 6.



We drop the 5 from the code and put 6 at the back: $(2, 3, 4, 7, 1, 6)$.

We have iterated all the way through the code and the two numbers missing are 5 and 8. So we connect them.



And we are done.

(2) Determine which trees have Prüfer codes that have distinct values in all positions.

Solution: The number of positions in a Prüfer code is two less than the number of vertices in the corresponding tree. So if a Prüfer code has distinct values in all positions, two vertex labels do not appear and the remaining labels appear only once. Observe that the degree of a vertex in a labeled tree is one more than the number of times the label of that vertex appears in the corresponding Prüfer code. Therefore, a tree with a distinct-valued Prüfer code has exactly two vertices of degree 1 and all other vertices have degree 2.

Now, let T be a tree with exactly two leaves and all other vertices having degree 2. Let P be the unique path in T from one leaf of T to the other. Then P is a subgraph of T. In fact, P is all of T, since the existence of a vertex of T that is not in P would change the degree of at least one vertex of P. Thus T must be a path.

(3) Let G be a connected graph which is not a tree and let C be a cycle in G. Prove that the complement of any spanning tree of G contains at least one edge of C.

Solution: Let T be a spanning tree of G. Let C be a cycle in G and assume that $\overline{T}$ contains no edge of C. Then T necessarily contains every edge of C. As T is a tree and therefore acyclic, we have a contradiction.

(4) Suppose a graph G is formed by taking two disjoint connected graphs G_1 and G_2 and identifying a vertex in G_1 with a vertex in G_2. Show that $\tau(G) = \tau(G_1)\tau(G_2)$.

Solution: Let G be the graph that results from identifying the vertex $v_1 \in V(G_1)$ with the vertex $v_2 \in V(G_2)$. Let $\mathcal{T}(G)$ be the *set* of all spanning trees of G. Similarly define $\mathcal{T}(G_1)$ and $\mathcal{T}(G_2)$. Assume that G_1 has n vertices and G_2 has m vertices. Then G contains $n + m - 1$ vertices.

Let $\phi : \mathcal{T}(G) \to \mathcal{T}(G_1) \times \mathcal{T}(G_2)$ send a spanning tree T of G to the ordered pair of subgraphs (T_1, T_2), where T_1 is the subgraph of G_1 induced by the edges of T that lie in G_1 *and* T_2 is the subgraph of G_2 induced by the edges of T that lie in G_2.

The graph T_1 is truly a spanning tree of G_1: that it is acyclic and connected follows from the fact that T is acyclic and connected. Similarly for T_2. Furthermore, the inverse image of a pair $(T_1, T_2) \in \mathcal{T}(G_1) \times \mathcal{T}(G_2)$ is a connected subgraph of G containing $n - 1 + m - 1 = n + m - 2 = (n + m - 1) - 1$ vertices, i.e., a spanning tree of G.

The map ϕ is bijective since a tree is completely defined by its edges and $E(G) = E(G_1) \cup E(G_2)$. Hence

$$|\mathcal{T}(G)| = |\mathcal{T}(G_1) \times \mathcal{T}(G_2)|.$$

Which implies that

$$\tau(G) = \tau(G_1)\tau(G_2).$$

(5) Assume the graph G has two components G_1 and G_2. Show there is a labeling of the vertices of G such that the adjacency matrix of G has the form

$$\mathbf{A}(G) = \begin{pmatrix} \mathbf{A}(G_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}(G_2) \end{pmatrix}.$$

Solution: Suppose that the graph G_1 contains p vertices and the graph G_2 contains q vertices. Then G has $p+q$ vertices.

Assign G_1 the vertex label $V(G_1) = \{u_1, \cdots u_p\}$ and G_2 the vertex label $V(G_2) = \{u_{p+1}, \cdots u_{p+q}\}$. Then the vertex label $V(G) = \{u_1, \cdots u_{p+q}\}$ induces an adjacency matrix for G with the required form.

(6) An *m-fold path*, mP_n, is formed from P_n by replacing each edge with a multiple edge of multiplicity m. An *m-fold cycle*, mC_n, is formed from C_n by replacing each edge with a multiple edge of multiplicity m.

(a) Find $\tau(mP_n)$
(b) Find $\tau(mC_n)$

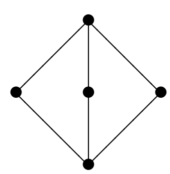
Solution: For part (a), there are m choices of edge for each edge of the underlying path, giving a total of m^{n-1} spanning trees.

For part (b), we observe that there are n ways to reduce the problem to that of part (a), giving a total of nm^{n-1} spanning trees.

(7) Find $\tau(K_{2,3})$.

Solution: There are, of course, many ways to go about this. One way is to note that $K_{2,3}$ has 5 vertices and 6 edges. Any spanning tree would have exactly four edges and there are $\binom{6}{2} = 15$ ways to remove two edges.

But this is clearly an over counting, since removing just any two edges doesn't work. You can see from the following representation of $\tau(K_{2,3})$ that the pairs of edges that don't produce a tree are those incident to the same vertex in the bipartition set of order 3.



There are 3 such pairs. Hence, $\tau(K_{2,3}) = 15 - 3 = 12$.

(8) Use the Matrix-Tree Formula to compute $\tau(K_{3,n})$.

Solution: The degree matrix of $K_{3,n}$ is the diagonal matrix where the (i,i) entry is the degree of the i^{th} vertex. Hence, with respect to an appropriate labeling, $D(K_{3,n})$ is the $(n+3) \times (n+3)$ matrix given by

$$D(K_{3,n}) = \begin{pmatrix} n & & & & & & \\ & n & & & & & \\ & & n & & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & \ddots & \\ & & & & & & 3 \end{pmatrix}$$

The adjacency matrix has as its (i,j) entry the number of edges between the i^{th} and j^{th} vertices. Therefore, with respect to the same vertex labeling used for the degree matrix, we have

$$A(K_{3,n}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & & & & \\ 1 & 1 & 1 & & & & \\ \vdots & \vdots & \vdots & & & \mathbf{0} & \\ 1 & 1 & 1 & & & & \end{pmatrix}$$

Therefore,

$$T(G) = D(K_{3,n}) - A(K_{3,n}) = \begin{pmatrix} n & & & -1 & -1 & \dots & -1 \\ & n & & -1 & -1 & \dots & -1 \\ & & n & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & 3 & & & \\ -1 & -1 & -1 & & 3 & & \\ \vdots & \vdots & \vdots & & & \ddots & \\ -1 & -1 & -1 & & & & 3 \end{pmatrix}$$

By the Matrix-Tree Theorem, $\tau(K_{3,n})$ is equal to any cofactor of $T(G)$. Let's compute the $(1,1)$ cofactor. To do so we remove the first row and first column from $T(G)$ and take the determinant of the resulting matrix (and then multiply by $(-1)^2 = 1$).

So, we have

$$\tau(K_{3,n}) = \det \begin{pmatrix} n & & -1 & -1 & \dots & -1 \\ & n & -1 & -1 & \dots & -1 \\ -1 & -1 & 3 & & & \\ -1 & -1 & & 3 & & \\ \vdots & \vdots & & & \ddots & \\ -1 & -1 & & & & 3 \end{pmatrix}$$

Subtracting the first column from the second column gives

$$\tau(K_{3,n}) = \det \begin{pmatrix} n & -n & -1 & -1 & \dots & -1 \\ & n & -1 & -1 & \dots & -1 \\ -1 & 0 & 3 & & & \\ -1 & 0 & & 3 & & \\ \vdots & \vdots & & & \ddots & \\ -1 & 0 & & & & 3 \end{pmatrix}$$

We then multiply the first column by 3.

$$\tau(K_{3,n}) = \frac{1}{3} \det \begin{pmatrix} 3n & -n & -1 & -1 & \dots & -1 \\ & n & -1 & -1 & \dots & -1 \\ -3 & 0 & 3 & & & \\ -3 & 0 & & 3 & & \\ \vdots & \vdots & & & \ddots & \\ -3 & 0 & & & & 3 \end{pmatrix}$$

Finally, by adding columns $2, \dots, n+2$ to the first column we get

$$\begin{aligned} \tau(K_{3,n}) =& \frac{1}{3} \det \begin{pmatrix} n & -n & -1 & -1 & \dots & -1 \\ & n & -1 & -1 & \dots & -1 \\ 0 & 0 & 3 & & & \\ 0 & 0 & & 3 & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & & & & 3 \end{pmatrix} \\ =& n^2 3^{n-1} \end{aligned}$$