# §2.2: FIRST ORDER LINEAR, II

This handout describes a much simplified version of the text's 'Euler-Lagrange Two Stage Method'or 'Variation of Parameters'. It is called the METHOD OF UNDETERMINED COEFFICIENTS. (Sorry for all the terminology. You will need to know this name; it comes up again in Math 5A.)

Instead of just telling you an algorithm; I will explain an idea that ties some stuff we've done so far together. Consider the following two examples of ODEs:

$$y' + y = t \qquad \text{and} \qquad y' + y = 2\sin(t).$$

If we apply the technique of direction fields we already learned, we see that neither one has any equilibrium solutions $y =$ constant. For the first, the isoclines are all straight line parallel to $y = t - 1$. Above the line $y = t - 1$, the slopes are less than $1$, and the concavity is 'up'. Below the line $y = t - 1$, the slopes are all greater than $1$ and the concavity is 'down.' See the top of Figure 1 for the computer generated picture. If that makes you wonder if the isocline $y = t - 1$ actually *is* a solution, congratulations. You can easily now check that $y = t - 1$ actually is a solution. Furthermore, the other solutions seems to be drawn towards it as $t$ increases.

The second ODE has no equilibrium solutions $y =$ constant, either. Algebra shows that $y = \sin(t) - \cos(t)$ is a solution, and again the other solutions seem to be attracted to it as $t$ increases, see the lower half of Figure 1.

To see why this is happening, we need to turn from the geometry to algebra. Think of the left side of the ODE as an operation $L$ that we do on function $y(t)$. So the input is $y$ and the output is $L(y) = y' + y$. We want to know what input $y$ makes the output $L(y) = t$. Since the operations of taking derivatives and adding converts polynomials to polynomials, it follows that if the output $L(y)$ is a first degree polynomial, so was the input. So $y = A + Bt$ for some $A$ and $B$. (These are the 'undetermined coefficients' of the method.) Then for this $y$, $L(y) = (A + B) + Bt$, so if $L(y) = t$, then $B = 1$ and $A = -1$, i.e. $y = t - 1$.

In the second ODE, it is clear that if the output is a trig function, $\sin(\mu t)$ or $\cos(\mu t)$ for some number $\mu$, then the input is some combination $A\sin(\mu t) + B\cos(\mu t)$ (We need both because taking derivatives

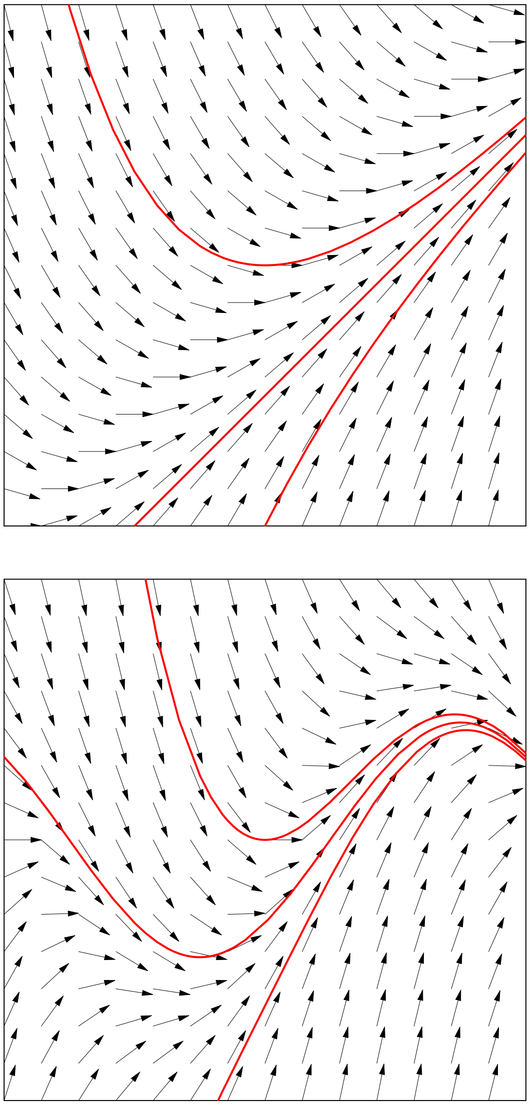

FIGURE 1

swaps sine and cosine.) Using this idea with $\mu = 1$, we deduce that $A = 1$ and $B = -1$. So $\sin(t) - \cos(t)$ is a solution, and this is how I got it above. (For comparison, try to solve this ODE by integrating factors.)

Taking derivatives and adding just multiplies exponential functions $\exp(\lambda t)$ by a scalar, so if we want $L(y) = \exp(\lambda t)$, then $y =$

$A\exp(\lambda t)$ for some $A$. For example, using this with the ODE

$$y' + y = \exp(-2t)$$

we see that $y(t) = A\exp(-2t)$ for some $A$, and plugging in and solving gives $A = -1$. In other words, $-\exp(-2t)$ is a solution to the ODE.

Algebraically what we have found is called a PARTICULAR SOLUTION $y_p(t)$. This terminology comes from the fact that there is no unknown constant of integration $+C$, so we can not yet solve an IVP. Physically or geometrically it is called the STEADY STATE SOLUTION. This is because, as Figure 1 seems to imply, the other solutions seem to be drawn towards the steady state. It is not an equilibrium because it is not constant, but it is the next best thing.

How do we find the general solution $y(t)$? Algebraically if $L(y) = f(t)$ (here we are writing the right hand side of the equation more generally), and also $L(y_p) = f(t)$, then

$$L(y - y_p) = L(y) - L(y_p) = f(t) - f(t) = 0.$$

In other words, $y(t) - y_p(t)$ is a solution to a different, much easier equation

$$y' + y = 0 \qquad \text{or} \qquad y' = -y.$$

This new ODE has solution $y_h = C\exp(-t)$, called the HOMOGENEOUS SOLUTION in algebra language. We have

$$y - y_p = y_h \qquad \text{so} \qquad y = y_p + y_h.$$

In physical or geometric language, $y_h$ is called the TRANSIENT SOLUTION, because of the exponential decay as $t$ increases.

In summary, in slightly more general language, for the CONSTANT COEFFICIENT FIRST ORDER LINEAR ODE

$$L(y) = y' + ay = f(t)$$

the GENERAL SOLUTION is

$$y = y_p + y_h, \qquad \text{where} \qquad y_h = C\exp(-at)$$

is the solution to the homogeneous equation $y' + ay = 0$, and $y_p$ is a particular solution. This can be found through the method of undetermined coefficients if the right side $f(t)$ is a polynomial, an exponential function, or a trigonometric function.

(i) Solve the following ODEs by the method of undetermined coefficients. Write the general solution as a particular solution

$y_p$ plus a solution $y_h$ to the homogeneous equation. Check your answers.

$$y' + 2y = t + 1 \tag{1}$$

$$y' - y = 2t + 3 \tag{2}$$

$$y' + y = t^2 \tag{3}$$

$$y' - y = 1 \tag{4}$$

$$y' - 2y = \sin(t) \tag{5}$$

$$y' - y = 2\sin(t) \tag{6}$$

$$y' + y = \sin(2t) \tag{7}$$

$$y' + y = \exp(2t) \tag{8}$$

$$y' - y = \exp(-t) \tag{9}$$

$$y' + y = \exp(t) \tag{10}$$

(ii) In the previous question you solved ten ODEs. Now write down the ten ODEs in scrambled order on one page, and the ten solutions also in scrambled order on another page. You know the general solution to the homogeneous equation 'by inspection'. And you know what the particular solution should look like by the method of undetermined coefficients. Use this to match ODEs to solutions. (This would make a very good multiple choice exam question, hint hint.)

(iii) Try to solve

$$y' + y = t + \exp(t)$$

by undetermined coefficients.