§1.2: DIRECTION FIELDS

In Math 3B, we think a lot about finding an unknown function $F(t)$ whose derivative *is* known, some given function $f(t)$. One approach is geometric. At each point in the plane, we know the slope of the function $F(t)$, so we can plot a little segment with that slope, and cover the whole plane. This gives a picture of what the graph of $f(t)$ should look like. The top of Figure 1 shows this for $f(t) = t$. Convince yourself of this: the slopes are large when the t coordinate is large and small when the t coordinate is small. The slopes do not depend on the y coordinate at all.

We can see solutions by following the slopes. They are parabolas just as we expect, since $F(t) = t^2/2 +$ a constant. This idea occurs in Math 3B but is not emphasized.

In this course, the derivative of the unknown function depends on both the t and y coordinates. The same idea as above gives the DIRECTION FIELD or SLOPE FIELD for the differential equation. In general it does not have the symmetry under vertical translation that the 3B examples had, so the problems are harder now. The bottom of Figure 1 shows the direction field for

$$y' = \frac{-t}{y}$$

Computers are helpful in drawing the direction field, but we will also make sketches by hand, for several reasons

1. (1) The process of thinking about what the picture looks like will call our attention to various features of the direction field, things we might overlook.
2. (2) The computer generated picture may miss important features, if they fall outside the viewing range selected.
3. (3) The computer generated picture may miss important features, if the wrong scale is selected

In making a sketch of the direction field for $y' = f(t, y)$, consider the following:

- (i) Are there any constant solutions $y = c$? This happens exactly when $y' = 0$ for *all* t, ie when there is a constant c so that $f(t, c) = 0$ for *all* t. Constant solutions represent physical equilibriums, so they are interesting. If we already know about the equilibrium based on our physical intuition, the

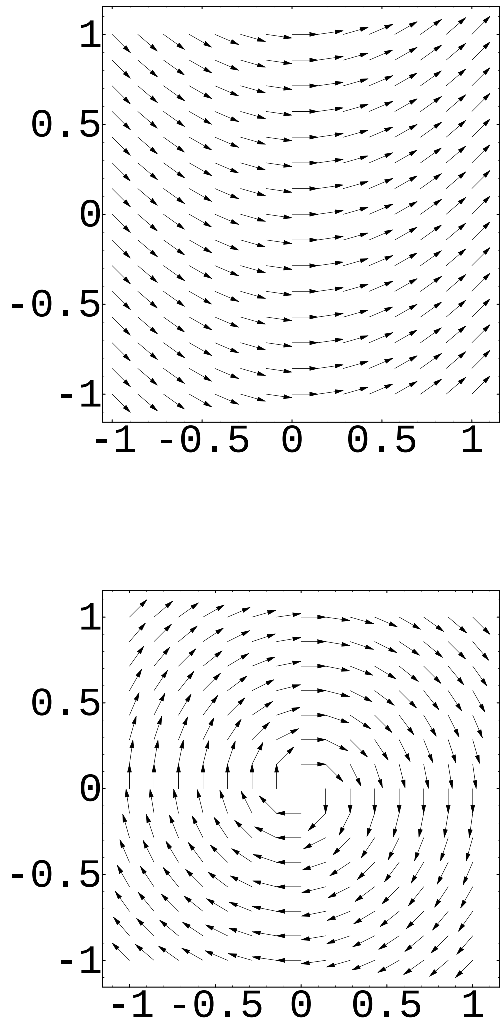

FIGURE 1

presence in the direction field reassures us the model is correct. We might also find equilibria we had not previously realized existed. In the ODE $y' = y$, the line $y = 0$ represents a constant solution.

(ii) The 'Fundamental Existence and Uniqueness Theorem'says there is exactly one solution through a point (t_0, y_0) if certain conditions are met. In particular f needs to be continuous at (t_0, y_0). In homework type examples what can easily go wrong is division by 0, or taking square root or log of a negative number. So you want to find the points in the plane, if any, where f is undefined. These are 'bad points'. In the

ODE $y' = -t/y$, there may be no solution or more than one for points where $y = 0$, the t axis.

(iii) Where is $f(t, y) = 0$? this is typically a curve in the plane. Any solution $y(t)$ will have to cross this with a horizontal tangent line. Plot this curve. Notice this is *not* the same as what you considered in (i). In the ODE $y' = -t/y$, the relevant curve is the line $t = 0$, the y-axis. Notice this is *not* an equilibrium solution. This curve is an example of a ISOCLINE. One could look at others, for example all the points in the plane where the solution has slope $+1$ or slope -1.

(iv) The curve in (iii) divides the plane into regions. In each we have $f(t, y) > 0$ always or $f(t, y) < 0$ always. So in each region the solutions $y(t)$ are always increasing or always decreasing. Label which is which. In the ODE $y' = -t/y$, the solutions are increasing in the second and fourth quadrants, and decreasing in the first and third.

(v) If you found one or more equilibrium solutions in step (i), you should wonder whether solutions near an equilibrium get closer to, or move further away from the equilibrium. This will have a physical meaning which will be important.

Sketch direction fields using the above criteria for the following ODEs:

$$y' = t - y \tag{1}$$

$$y' = y/t \tag{2}$$

$$y' = t^2 + y^2 \tag{3}$$

$$y' = 1/t \tag{4}$$

$$y' = y(2 - y) \tag{5}$$

$$y' = t + 2y \tag{6}$$

$$y' = ty \tag{7}$$

$$y' = ty^2 \tag{8}$$

$$y' = -\sqrt{y} \tag{9}$$

$$y' = (y - 1)(y - 2) \tag{10}$$

$$y' = \frac{yt - y}{t} \tag{11}$$

$$y' = \frac{t - y}{2t + y} \tag{12}$$

$$y' = t^2 - y^2 \tag{13}$$