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A Generalization of p-Rings

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Abstract

Let R be a ring with Jacobson ideal J and center C. McCoy and Montgomery introduced the concept of a p-ring (p prime) as a ring R of characteristic p such that $x^p = x$ for all x in R. Thus, Boolean rings are simply 2-rings ($p = 2$). It readily follows that a p-ring (p prime) is simply a ring R of prime characteristic p such that $R \subseteq N + E_p$, where $N = \{0\}$ and $E_p = \{x \in R : x^p = x\}$. With this as motivation, we define a generalized p-ring to be a ring of prime characteristic p such that $R \setminus (J \cup C) \subseteq N + E_p$, where N denotes the set of nilpotents of R (and E_p is as above). The commutativity behavior of these rings is considered.

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1 Introduction and preliminaries

McCoy and Montgomery [2] introduced the concept of a p-ring (p prime) as a ring R of prime characteristic p such that $x^p = x$ for all x in R. This is equivalent to saying that R is of prime characteristic p and

$$R \subseteq N + E_p,\ N = \{0\},\ E_p = \{x \in R : x^p = x\}. \tag{1}$$

With this as motivation, we define a generalized p-ring as follows:

Definition 1. *A* <u>*generalized p-ring*</u> *is a ring R of prime characteristic p such that*

$$R \setminus (J \cup C) \subseteq N + E_p, \quad N = N(R) \textit{ is the set of nilpotents of } R, \quad E_p = \{x \in R : x^p = x\} \qquad (2)$$

The class of generalized p-rings (p prime) is large and contains all commutative rings and all radical rings ($R = J$) as long as these are of prime characteristic p. It also contains all p-rings (p prime). However, a generalized p-ring is not necessarily commutative, as can be seen by taking

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}; \ 0, 1 \in GF(2) \right\}.$$

Indeed, R is a generalized 2-ring which is not commutative and not a 2-ring. In Theorem 4, we characterize all *commutative* generalized p-rings (p prime). In preparation for the proofs of the main theorems, we have the following two lemmas.

Lemma 1. *([1]) Suppose R is a ring in which every element x is central or "potent" in the sense that $x^k = x$ for some $k > 1$. Then R is commutative.*

Lemma 2. *Suppose R is a ring with central idempotents and suppose $x \in N$, N is the set of nilpotents. Suppose, further, that $ax - (ax)^n \in N$ for some $n > 1$. Then $ax \in N$.*

Proof. Since $ax - (ax)^n \in N$, $n > 1$, $(ax)^q = (ax)^{q+1} g(ax)$, $g(\lambda) \in \mathbb{Z}[\lambda]$. Let $e = ((ax)g(ax))^q$. Then $e^2 = e$, and hence

$$e = ee = e((ax)g(ax))^q = eat = aet.$$

So $e = aet = a^2et^2 = \ldots = a^k et^k$ for all positive integers k. Since $a \in N$, $a^k = 0$ for some $k \geq 1$, which implies that $e = a^k et^k = 0$. Thus, $(ax)^q = (ax)^q e = 0$, and hence $ax \in N$, which proves the lemma. □

2 Main results

Theorem 1. *Suppose R is a generalized p-ring (p prime) with identity and with central idempotents. Then*

$$(i)\ E_p \subseteq C, \ \textit{and}\ (ii)\ N \subseteq J \subseteq N \cup C.$$

Proof (i). Let $b \in E_p$ and let $r \in R$. Since $b^p = b$ (by definition of E_p), b^{p-1} is idempotent, which is central (by hypothesis) and hence

$$b^{p-1}(rb - br) = rb^p - b^p r = rb - br,$$

which implies that

$$(b^{p-1} - 1)(rb - br) = 0 \quad \text{for all } r \text{ in } R. \tag{3}$$

Since R is of *prime* characteristic p, an elementary number-theoretic result shows that (3) is equivalent to

$$(b+1)(b+2)\cdots(b+(p-1))(rb - br) = 0, \quad (r \in R). \tag{4}$$

Furthermore, since R is of *prime* characteristic p, we have:

$$b^p = b \text{ implies } (b+1)^p = b+1,$$

and hence the above argument may be repeated with b replaced by $b+1$ throughout. Thus (4) now yields

$$(b+2)(b+3)\cdots(b+(p-1))(b+p)(r(b+1) - (b+1)r) = 0,$$

and hence

$$b(b+2)(b+3)\cdots(b+(p-1))(rb - br) = 0. \tag{5}$$

Subtracting (5) from (4), we obtain

$$1 \cdot (b+2)(b+3)\cdots(b+(p-1))(rb - br) = 0. \tag{6}$$

Repeating this argument, where b is replaced by $b+1$ again throughout, we see that

$$1 \cdot (b+3)(b+4)\cdots(b+(p-1))(b+p)(rb - br) = 0,$$

and hence

$$1 \cdot b(b+3)(b+4)\cdots(b+(p-1))(rb - br) = 0. \tag{7}$$

Subtracting (7) from (6), we obtain

$$1 \cdot 2 \cdot (b+3)(b+4)\cdots(b+(p-1))(rb - br) = 0.$$

Continuing this process, we eventually obtain

$$(p-1)!(rb - br) = 0 \text{ for all } r \text{ in } R. \tag{8}$$

Since $(p-1)!$ is relatively prime to the prime characteristic p of R, (8) yields $rb - br = 0$ for all r in R, and hence b is central, which proves part (i).

(ii) Let $a \in N$, $x \in R$. If $ax \in J$, then ax is r.q.r. Also, if $ax \in C$, then $ax \in N$ and hence again ax is r.q.r. Now suppose that $ax \notin (J \cup C)$. Then, by (2),

$$ax = a_0 + b_0;\ a_0 \in N,\ b_0^p = b_0, \text{ and } b_0 \in C, \text{ by part (i).} \tag{9}$$

Since $b_0 \in C$, $[b_0, ax] = 0$, and hence $[b_0, a_0] = 0$. So, by (9), $[ax - a_0, a_0] = 0$, which implies $[ax, a_0] = 0$. Then, by (9) again,

$$ax - a_0 = (ax - a_0)^p,\ [ax, a_0] = 0.$$

Since R is of prime characteristic p and ax commutes with a_0, $(ax - a_0)^p = (ax)^p - a_0^p$ and hence $(ax - a_0) = (ax)^p - a_0^p$. So $ax - (ax)^p = a_0 - a_0^p \in N$, which implies, by Lemma 2, $ax \in N$. Since $ax \in N$, ax is r.q.r. for all $x \in R$, and hence $a \in J$. So

$$N \subseteq J \tag{10}$$

Next, we prove that $J \subseteq N \cup C$. To prove this, let $j \in J \setminus C$. Then, $1 + j \notin (J \cup C)$, and hence by (2)

$$1 + j = a + b,\ a \in N,\ (b^p = b, \text{ and hence } b \in C, \text{ by part (i)).} \tag{11}$$

Since $b \in C$, $[1 + j - a, a] = 0$ which implies $[1 + j, a] = 0$.
So, $1 + j - a = b = b^p = (1 + j - a)^p = (1 + j)^p - a^p$ (since $1 + j$ commutes with a), which implies

$$1 + j - (1 + j)^p = a - a^p \in N. \tag{12}$$

So $1 + j - (1 + j^p) \in N$, and hence $j - j^p \in N$. Thus,

$$j = j(1 - j^{p-1})(1 - j^{p-1})^{-1} = (j - j^p)(1 - j^{p-1})^{-1} \in N,$$

since $j - j^p \in N$. Hence, $j \in N$. This proves part (ii). □

Theorem 2. *Under the hypotheses of Theorem 1, we have (i) N is an ideal and (ii) R/N is commutative. Thus, the commutator ideal of R is nil.*

Proof. (i) Let $a \in N$, $b \in N$. Then, by Theorem 1 (ii), $a \in J$, $b \in J$, and hence $a - b \in J$. Since $j \subseteq N \cup C$ (Theorem 1 (ii)) we have $a - b \in N$ or $a - b \in C$. If $a - b \in C$, then a commutes with b, and hence $a - b \in N$. So in any case $a - b \in N$. Next, suppose $a \in N$, $x \in R$. Then $a \in J$ (Theorem 1 (ii)), $x \in R$, and hence $ax \in J \subseteq N \cup C$ (by Theorem 1 (ii)). So $ax \in N$ or $ax \in C$. If $ax \in C$, then $(ax)^k = a^k x^k$ for all $k \geq 1$, and hence $ax \in N$ (since $a \in N$). So in any case $ax \in N$. Similarly $xa \in N$, which proves

$$N \text{ is an ideal.} \tag{13}$$

(ii) Since $N \subseteq J \subseteq N \cup C$ (Theorem 1 (ii)), it follows that

$$N \cup C \subseteq J \cup C \subseteq (N \cup C) \cup C = N \cup C,$$

and hence $J \cup C = N \cup C$. Therefore, by (2),

$$\forall x \in R \setminus (N \cup C),\ x = a + b,\ a \in N,\ b^p = b. \tag{14}$$

Since (14) is trivially satisfied if $x \in N$, we conclude that

$$\forall x \in R \setminus C,\ x = a + b,\ a \in N,\ b^p = b. \tag{15}$$

Combining (13) and (15), we conclude that every element of R/N is central or potent ($\bar{x}^p = \bar{x}$). Therefore, by Lemma 1, R/N is commutative, and thus the commutator ideal of R is nil. This completes the proof. □

In the following we obtain our first commutativity theorem of the ground ring R by adding one additional hypothesis.

Theorem 3. *Suppose R is a generalized p-ring (p prime) with identity and with central idempotents. Suppose, further, that $N \cap J$ is commutative. Then R is commutative.*

Proof. By Theorem 1 (ii), $N \subseteq J \subseteq N \cup C$, and hence (as shown in the proof of that theorem), $J \cup C = N \cup C$. Hence (see the proof of (15)) we have

$$\forall x \in R \setminus C,\ x = a + b, a \in N,\ b^p = b,\ b \in C \text{ (by Theorem 1(i))}. \tag{16}$$

Suppose that, for some $x, y \in R$, $[x, y] \neq 0$. Then $x \notin C$ and $y \notin C$, which implies by (16) that

$$[x, y] = [a + b, a' + b'],\ a, a' \in N,\ b^p = b,\ (b')^p = b'. \tag{17}$$

Moreover, in (17), $b \in C$, $b' \in C$, by Theorem 1 (i). So (17) readily implies

$$[x, y] = [a, a'],\ (a, a' \in N). \tag{18}$$

Since $N \subseteq J$ (Theorem 1 (ii)), $N \cap J = N$, and hence N is commutative (since, by hypothesis, $N \cap J$ is commutative). Combining this fact with (18), we conclude that $[x, y] = 0$, contradiction. This proves the theorem. □

Corollary 1. *A generalized p-ring (p prime) with identity and with central idempotents and commuting nilpotents is commutative.*

In our final theorem, we delete the hypothesis that R has an identity and at the same time strengthen the hypothesis that $N \cap J$ is commutative.

Theorem 4. *Suppose R is any generalized p-ring (p prime), not necessarily with identity. Suppose that the idempotents of R are central and J is commutative. Then R is commutative (and conversely).*

Proof. Case 1. $1 \in R$. Then by Theorem 3, R is commutative. For the general case, where we no longer assume that R has an identity, we distinguish two cases.

Case A. $E_p = \{0\}$. In this case, we have $R = N \cup J \cup C$ (see (2)). Let $a \in N$, $x \in R$. If $ax \in N$, then ax is r.q.r. Also, if $ax \in J$, then ax is r.q.r. Finally, if $ax \in C$, then $ax \in N$, and hence again ax is r.q.r. So ax is r.q.r. for all $x \in R$, and hence $N \subseteq J$, which implies that $R = J \cup C$. Since, by hypothesis, J is commutative, R is commutative (if $E_p = \{0\}$).

Next, consider the case $E_p \neq \{0\}$. Let $b \in E_p$, $b \neq 0$. Then $b^p = b$, and hence $e = b^{p-1}$ is a nonzero central idempotent (recall that, by hypothesis, all idempotents are central). It can be verified that eR is a ring *with identity* e which in fact satisfies all the hypotheses imposed on R. In verifying this, recall that $J(eR) \subseteq J(R)$, and hence $J(eR)$ is commutative, since $J(R)$ is commutative. Therefore, by case 1, eR is commutative. Next, we prove that

$$E_p \subseteq C \text{ (the center of } R). \tag{19}$$

(Note that Theorem 1 (i) no longer applies here, since we are not assuming that $1 \in R$). To prove (19), let $b \in E_p$, $y \in R$. Recall that $e = b^{p-1}$ is in the center of R. Since eR is commutative,

$$0 = [eb, ey] = ebey - eyeb = eby - yeb = b^p y - yb^p = by - yb,$$

and hence $[b, y] = 0$ for all $y \in R$, which proves (19).

We claim that

$$N \subseteq J \tag{20}$$

(Note again that Theorem 1 (i) no longer applies here, since we are not assuming that $1 \in R$.) To prove (20), let $a \in N$, $x \in R$. If $ax \in J$ or $ax \in C$, then (as we saw above), ax is r.q.r. Suppose $ax \notin (J \cup C)$. Then, by (2),

$$ax = a_0 + b_0;\ a_0 \in N,\ b_0^p = b_0, \text{ and hence } b_0 \in C, \text{ by (19)}. \tag{21}$$

Thus, $ax - a_0 = (ax - a_0)^p$ and $[ax, a_0] = 0$ (since $b_0 \in C$), which readily implies that $ax - (ax)^p \in N$. Hence, by Lemma 2, $ax \in N$, and thus ax is r.q.r. for all $x \in R$. So $a \in J$, proving (20).

To complete the proof , note that N is commutative (since J is commutative; see (20)). Assume, for the moment, that x_1, x_2 are not in $(J \cup C)$. Then,

$$x_1 = a_1 + b_1; x_2 = a_2 + b_2;\ a_1,\ a_2 \in N,\ b_1^p = b_1,\ b_2^p = b_2. \tag{22}$$

Combining (22) and (19), we see that

$$x_1 = a_1 + b_1; x_2 = a_2 + b_2;\ a_1,\ a_2 \in N,\ b_1,\ b_2 \in C, \tag{23}$$

and hence

$$[x_1, x_2] = [a_1 + b_1, a_2 + b_2] = [a_1, a_2] = 0 \text{ (since } N \text{ is commutative).}$$

Thus, in this present case, $[x_1, x_2] = 0$. The case where $x_1 \in (J \cup C)$ or $x_2 \in (J \cup C)$ readily yields $[x_1, x_2] = 0$ (since J is commutative and $N \subseteq J$).

Hence, R is commutative, and the theorem is proved. □

The following corollary was first proved in [2].

Corollary 2. *A p-ring R is commutative.*

Proof. It is readily seen that in a p-ring R all idempotents are central and $J = \{0\}$. □

Related work appears in [3].

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