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Results in Mathematics

On Commutativity of Semiperiodic Rings

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Abstract. Let R be a ring with center Z, Jacobson radical J, and set N of all nilpotent elements. Call R semiperiodic if for each $x \in R \setminus (J \cup Z)$, there exist positive integers m, n of opposite parity such that $x^n - x^m \in N$. We investigate commutativity of semiperiodic rings, and we provide noncommutative examples.

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1. Introduction

Let R be a ring with center $Z = Z(R)$, Jacobson radical $J = J(R)$, and set $N = N(R)$ of all nilpotent elements; and let $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the ring of integers and the set of positive integers. Define R to be periodic if for each $x \in R$, there exist distinct positive integers m, n such that $x^n = x^m$. It is known that R must be periodic if each $x \in R$ satisfies the Chacron criterion [5]:

$$\text{there exists} \quad m \in \mathbb{Z}^+ \quad \text{and} \quad p(t) \in \mathbb{Z}[t] \quad \text{such that} \quad x^m = x^{m+1}p(x)\,. \qquad (*)$$

It follows that R is periodic if for each $x \in R$ there exist distinct $m, n \in \mathbb{Z}^+$ for which $x^n - x^m \in N$.

In this paper we study rings in which an appropriate subset of elements of R satisfy the Chacron criterion. Specifically, we define R to be semiperiodic if for each $x \in R \setminus (J \cup Z)$ there exist $m, n \in \mathbb{Z}^+$, of opposite parity, such that $x^n - x^m \in N$. Clearly, the class of semiperiodic rings contains all commutative rings, all Jacobson radical rings, and certain non-nil periodic rings; and it contains the generalized periodic-like rings discussed in [4]. We shall be principally concerned with commutativity and near-commutativity of semiperiodic rings.

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2. Preliminaries

We begin with a bit of additional terminology. A ring R is called reduced if $N = \{0\}$, and R is called normal if all idempotents are central. An element $x \in R$ is periodic if there exist distinct $m, n \in \mathbb{Z}^+$ for which $x^n = x^m$; and x is potent if there exists $n \in \mathbb{Z}^+$, $n > 1$, such that $x^n = x$. It is easy to show that if R is reduced, every periodic element is potent.

As usual, if $x, y \in R$, the symbol $[x, y]$ represents the commutator $xy - yx$; and extended commutators $[x, y]_k$, $k \geq 1$, are defined inductively by taking $[x, y]_1 = [x, y]$ and $[x, y]_k = [[x, y]_{k-1}, y]$. It is easily verified that

$$[x, y]_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} y^i x y^{k-i} \,. \tag{2.1}$$

The symbol $\mathcal{C}(R)$ denotes the commutator ideal of R, and $\langle x \rangle$ denotes the subring generated by x. Finally, the symbol $((m, n))$ represents an ordered pair of positive integers of opposite parity.

We now state two lemmas which apply to rings which are not necessarily semiperiodic, followed by several lemmas dealing with elementary properties of semiperiodic rings.

Lemma 2.1 ([2, Lemma 2]). *If R is any ring in which each element is central or potent, then R is commutative.*

Lemma 2.2. *If R is a ring containing an ideal I such that both I and R/I are commutative, then the commutator ideal $\mathcal{C}(R)$ is nil and N is an ideal.*

Proof. Since R/I is commutative, $[x, y] \in I$ for all $x, y \in R$; and since I is commutative, R satisfies the identity $[[x, y], [u, v]] = 0$. The conclusion now follows by Theorem 1 of [1]. □

Lemma 2.3. *Let R be any semiperiodic ring.*

(i) *Every epimorphic image of R is semiperiodic, and every ideal of R is semiperiodic.*
(ii) *If e is any idempotent with additive order not a power of 2, then $e \in Z$.*
(iii) *If $x \notin J \cup Z$, there exists $q \in \mathbb{Z}^+$ and $g(t) \in t\mathbb{Z}[t]$ such that $e = g(x)$ is idempotent and $x^q = x^q e$.*

Proof. (i) This is immediate, once we recall that if $\sigma\colon R \to S$ is a ring epimorphism, then $\sigma(J(R)) \subseteq J(S)$, and that $J(I) = I \cap J(R)$ for every ideal I of R.
(ii) If e is a noncentral idempotent, then $-e \notin J \cup Z$, hence there exists $((m, n))$ for which $(-e)^n - (e)^m \in N$. Since m and n are of opposite parity, $2e \in N$ and hence there exists $k \in \mathbb{Z}^+$ such that $2^k e = 0$.
(iii) If $x \notin J \cup Z$, there exists $((m, n))$ such that $x^n - x^m \in N$ and therefore there exist $q \in \mathbb{Z}^+$ and $f(t) \in \mathbb{Z}[t]$ such that $x^q = x^{q+1} f(x)$. It is easily verified that $e = (xf(x))^q$ is an idempotent with $x^q = x^q e$. □

Lemma 2.4. *If R is a normal semiperiodic ring, then $N \subseteq J$.*

Proof. Let $a \in N$ with $a^k = 0$, and let $x \in R$. If $ax \in J$, then ax is right quasi-regular; and if $ax \in Z$, then ax is nilpotent and again ax is right quasi-regular. Suppose, then, that $ax \notin J \cup Z$, in which case Lemma 2.3 (iii) gives $q \in \mathbb{Z}^+$ and a central idempotent e of form ay such that $(ax)^q = (ax)^q e$. Since $e \in Z$, $e = e^2 = eay = aey = a^2ey^2 = \cdots = a^k ey^k = 0$; hence $(ax)^q = 0$ and ax is right quasi-regular. Thus $a \in J$. □

Lemma 2.5. *If R is a normal semiperiodic ring and $\sigma \colon R \to S$ is a ring epimorphism, then $N(S) \subseteq Z(S) \cup \sigma(J)$.*

Proof. Suppose $w \in N(S) \setminus (Z(S) \cup \sigma(J))$ with $w^q = 0$, and let $u \in \sigma^{-1}(w)$. Since $u \notin Z(R) \cup J$, there exists $k > 1$ such that $u - u^k \in N$; and by Lemma 2.4, $u - u^k \in J$. It follows that $(u - u^k) + u^{k-1}(u - u^k) + u^{2(k-1)}(u - u^k) + \cdots + u^{q(k-1)}(u - u^k) \in J$ – i.e., $u - u^{(q+1)(k-1)+1} \in J$. Thus, by applying σ, we get $w - w^{(q+1)(k-1)+1} = w \in \sigma(J)$ – a contradiction. □

Lemma 2.6. *If R is a semiperiodic ring with 1, then $J \subseteq N$ or $J \subseteq Z$.*

Proof. Suppose $j \in J \setminus Z$. Then $1 + j$ and $-1 + j$ are not in $J \cup Z$, hence there exist $((m_1, n_1))$ and $((m_2, n_2))$ such that

$$(1 + j)^{n_1} - (1 + j)^{m_1} \in N \quad \text{and} \quad (-1 + j)^{n_2} - (-1 + j)^{m_2} \in N\,.$$

It follows that

$$(m - n)j + j^2 f(j) \in N \quad \text{for some} \quad f(t) \in \mathbb{Z}[t] \tag{2.2}$$

and

$$2 + jg(j) \in N \quad \text{for some} \quad g(t) \in \mathbb{Z}[t]\,. \tag{2.3}$$

The latter condition implies that

$$2j + j^2 g(j) \in N\,; \tag{2.4}$$

and from (2.2) and (2.4), together with the fact that sums of commuting elements of N are in N, we get $h(t) \in \mathbb{Z}[t]$ such that $j + j^2 h(t) \in N$. Arguing as in the proof of Lemma 2.3(iii), we obtain $k \in \mathbb{Z}^+$ for which $j^k = j^k e$ for some idempotent $e \in J$. Thus $j^k = 0$.

We have now shown that $J \subseteq Z \cup N$. If $J \not\subseteq Z$ and $j \in J \setminus Z$, then $j \in N$. For any $c \in J \cap Z$, $j + c \in J \setminus Z$, hence $c = j + c - j$ is a difference of commuting elements of N, and therefore $c \in N$. Thus $J \subseteq Z$ or $J \subseteq N$. □

3. Near-commutativity theorems

Theorem 3.1. *If R is a normal semiperiodic ring, then R/J is commutative. If in addition J is commutative, then N is an ideal and R/N is commutative.*

Proof. Let $\bar{R} = R/J$. Since $N \subseteq J$ by Lemma 2.4, $\bar{R}$ has the property that

$$\text{for each} \quad x \in \bar{R}\,, \quad \text{either} \quad x \in Z(\bar{R}) \quad \text{or} \quad x^n = x^m \quad \text{for some} \quad ((m,n))\,. \tag{3.1}$$

Let S be a primitive image of $\bar{R}$. Since (3.1) is inherited by epimorphic images and subrings, and since 2×2 matrix rings over division rings do not satisfy (3.1), we conclude that S must be a division ring. But a division ring satisfying (3.1) has the property that each element is central or potent, hence is commutative by Lemma 2.1. Therefore, by the density theorem, $\bar{R}$ is commutative. If $J(R)$ is commutative, then by Lemma 2.2, N is an ideal and R/N is commutative. □

Corollary 3.2. *If R is a reduced semiperiodic ring with J commutative, then R is commutative.*

Proof. It is well-known that reduced rings are normal, hence the result follows at once from Theorem 3.1. □

Theorem 3.3. *If R is a normal semiperiodic ring with 1, then N is an ideal and R/N is commutative.*

Proof. Since $N \subseteq J$ by Lemma 2.4, it follows by Lemma 2.6 that $J = N$ or $J \subseteq Z$. In the first case, it is obvious that N is an ideal; and R/N is commutative because each of its elements is central or potent. In the second case the conclusion follows immediately by Theorem 3.1. □

4. Some commutativity theorems

A well-known theorem of Herstein [8] asserts that a periodic ring with $N \subseteq Z$ must be commutative. We begin this section by presenting similar theorems for semiperiodic rings.

Theorem 4.1. *If R is a normal semiperiodic ring with $J \subseteq Z$, then R is commutative.*

Proof. By Lemma 2.4, $N \subseteq J$ and therefore $N \subseteq Z$. Since $J \subseteq Z$, for each $x \in R \setminus Z$ there exists $((m,n))$ with $n > m$ for which $x^n - x^m \in N$. It follows easily that $x^{n-m+1} - x \in N \subseteq Z$, hence R is commutative by a well-known theorem of Herstein [7]. □

Corollary 4.2. *If R is a semiperiodic ring with 1 in which $N \subseteq Z$, then R is commutative.*

Proof. Since $N \subseteq Z$, R is normal; and by Lemma 2.6, $J \subseteq Z$. □

Our next theorem, which was a surprise, will be used in the proofs of the final two theorems in this section.

Theorem 4.3. *Let R be semiperiodic with 1. Then either R is commutative, or R is periodic and $(R,+)$ is a 2-group.*

Proof. Suppose that R is not commutative, so that by Corollary 4.2 we have an element $x \in N \setminus Z$. Then $-1 + x \notin J \cup Z$, hence we have $((m,n))$ for which $(-1+x)^n - (-1+x)^m \in N$. It follows that $2 + u \in N$ for some $u \in N$, so that 2 is the difference of two commuting nilpotent elements, hence $2 \in N$. Therefore, $(R,+)$ is a 2-group.

It is now clear that if $x \in R \setminus (J \cup Z)$, $\langle x \rangle$ is finite and hence x is periodic. By Lemma 2.6, $J \subseteq N$ or $J \subseteq Z$. If $J \subseteq Z$, each $y \in R \setminus Z$ is periodic; and if $x \in Z$, $x = x + y - y$ is a sum of commuting periodic elements. Thus every power of x has form $\sum n_i (x+y)^j y^k$, $n_i \in \mathbb{Z}$; and since there are only finitely many such sums, x is periodic. Hence in every case J is periodic.

Finally, suppose $x \in Z \setminus J$. Since R has 1, $R \neq J$; and since R is not commutative, $R \neq Z$. Thus there exists $y \in R \setminus (J \cup Z)$. Then either $x + y \in J$ or $x + y \in R \setminus (J \cup Z)$, so $x + y$ is periodic. Thus $x = x + y - y$ is a sum of commuting periodic elements, and we argue as above that x is periodic. We have now shown that R is periodic. □

Theorem 4.4. *If R is a reduced semiperiodic ring with $R \neq J$, then R is commutative.*

Proof. If $R = J \cup Z$, then $R = Z$ and we are finished. Assume that $R \neq J \cup Z$, in which case every element of $R \setminus (J \cup Z)$ is potent. Let a be any nonzero potent element, with $a^n = a$, $n > 1$ Then $e = a^{n-1}$ is a nonzero idempotent, necessarily central since reduced rings are normal. By Lemma 2.3 (i), eR is a semiperiodic ring with identity; hence by Lemma 2.6, $J(eR) \subseteq Z(eR)$. Thus, by Theorem 4.1, eR is commutative; and for each $x \in R$, $[ea, ex] = 0 = [ea, x] = [a, x]$. We have shown that all potent elements are central, so we cannot have $R \neq J \cup Z$. □

Theorem 4.5. *If R is a 2-torsion-free semiprime semiperiodic ring with $R \neq J$, then R is commutative.*

Proof. Since R is 2-torsion-free, R is normal by Lemma 2.3 (ii); hence $N \subseteq J$ by Lemma 2.4. If R is reduced, then R is commutative by Theorem 4.4. Assume, then, that R is not reduced. Recalling that in a semiprime ring $N \cap Z = \{0\}$, conclude that $N \not\subseteq Z$. Let $u \in N \setminus Z$ and suppose e is a nonzero idempotent. Then $-e+u \in R \setminus (J \cup Z)$, and there exists $((m,n))$ for which $(-e+u)^n - (-e+u)^m \in N$. It follows that $2e$ is a sum of commuting nilpotent elements, hence $2e \in N$ and $2^k e = 0$ for some $k \in \mathbb{Z}^+$, contradicting our hypotheses that R is 2-torsion-free. Thus, R has no nonzero idempotents; hence by Lemma 2.3 (iii) and our observation that $N \subseteq J$, we get $R = J \cup Z$, in which case $R = Z$. This contradicts our assumption that $N \not\subseteq Z$, so R must in fact be reduced. □

Theorem 4.6. *Let R be a semiperiodic ring with $R \neq J$. If both R and R/J are 2-torsion-free, then R is commutative.*

Proof. Since R is 2-torsion-free, R is normal; hence, if $J \subseteq Z$, R is commutative by Theorem 4.1. Suppose that $J \not\subseteq Z$, let $j \in J \setminus Z$, and let e be any nonzero idempotent. Then $-e + j \notin J \cup Z$, so there exists $((m,n))$ such that $(-e+j)^n -$

$(-e+j)^m \in N$. Therefore, $2e+u \in N$ for some $u \in J$, and hence $2^k e + v = 0$ for some $k \in \mathbb{Z}^+$ and $v \in J$.

Letting $\bar{e}$ be the image of e in R/J, we see that $2^k \bar{e} = 0$; and since R/J is 2-torsion-free, $\bar{e} = 0$. Thus $e \in J$, so that $e = 0$. As in the proof of Theorem 4.5, this yields a contradiction; hence $J \subseteq Z$ and we are finished. □

Theorem 4.7. *Let R be a semiperiodic ring with 1 which satisfies the following two conditions:*

(i) *For each $a \in N$ and $x \in R$, there exists $k \in \mathbb{Z}^+$ for which $[a,x]_k = 0$;*
(ii) *N is commutative.*

Then R is commutative.

Proof. We appropriate a method of proof used in [3]. Note first that if $e \in R$ is idempotent, the condition $[ex-exe, e]_k = 0$ is just the statement that $ex - exe = 0$. Similarly, $xe - exe = 0$, so R is normal. Therefore by Theorem 3.3, N is an ideal and $[x,y] \in N$ for all $x, y \in R$.

By Theorem 4.3, we may assume that R is periodic and $(R,+)$ is a 2-group, so that for each $x \in R$ the subring $R_x = \langle x \rangle$ is finite. The ring $R_x/N(R_x)$ is finite and reduced, hence a direct sum of finite fields of characteristic 2. It follows that there exists $s \in \mathbb{Z}^+$ such that

$$x^{2^{st}} - x \in N \quad \text{for all} \quad t \in \mathbb{Z}^+ . \tag{4.1}$$

Since $2R \subseteq N$, (ii) yields $2[a,x] = [a,2x] = 0$ for all $a \in N$ and $x \in R$. Suppose that $a \in N$ and x is such that $[a,x]_k = 0$, $k > 1$. Then $[[a,x],x]_{k-1} = 0$; and choosing $w \in \mathbb{Z}^+$ such that $2^w \geq k-1$, we have $[[a,x],x]_{2^w} = 0$. But by (2.1) and the fact that $2[a,x] = 0$, we have $[[a,x],x]_{2^w} = [[a,x],x^{2^w}]$, hence $[[a,x],x^{2^w}] = 0$. Thus $x^{2^{sw}} = (x^{2^w})^{2^{sw-w}}$ commutes with $[a,x]$; and by (ii) and (4.1), so does $x^{2^{sw}} - x$. It follows that $[[a,x],x] = 0$; therefore $[a, x^{2^s}] = 2^s x^{2^s - 1}[a,x] = 0$, and another appeal to (4.1) gives $[a,x] = 0$. We have now shown that $N \subseteq Z$, so R is commutative by Corollary 4.2. □

If we strengthen condition (ii), we can drop the hypothesis that R has 1. Specifically, we have:

Theorem 4.8. *Let R be a semiperiodic ring satisfying the following conditions:*

(i) *For each $a \in N$ and $x \in R$, there exists $k \in \mathbb{Z}^+$ for which $[a,x]_k = 0$;*
(ii)' *J is commutative.*

Then R is commutative.

Proof. Since (i) implies R is normal, Theorem 3.1 shows that $[x,y] \in N$ for all $x, y \in R$; and it follows from (i) that

$$\text{for each} \quad x, y \in R \quad \text{there exists} \quad k \in \mathbb{Z}^+ \quad \text{such that} \quad [x,y]_k = 0 . \tag{4.2}$$

Of course we need only show that subdirectly irreducible images of R are commutative.

Let S be any subdirectly irreducible image of R and let $\sigma\colon R \to S$ be an epimorphism. If S has no nonzero central idempotents, it follows from Lemma 2.3 (iii) that $\sigma(R\backslash(J\cup Z)) \subseteq N(S)$, so that $S = \sigma(J)\cup\sigma(Z)\cup N(S)$. But by Lemma 2.5, $N(S) \subseteq Z(S)\cup\sigma(J)$; hence $S = Z(S)\cup\sigma(J)$ and therefore S is commutative. Now suppose S has a nonzero central idempotent – i.e. S has 1. It follows from (4.2) that S inherits property (i), and by Lemma 2.5 $N(S)$ is commutative. Thus, S is commutative by Theorem 4.7. □

5. Examples

In this section we provide examples of noncommutative semiperiodic rings R with 1. Theorem 4.3 is very helpful; it says that $(R,+)$ is a 2-group and R is periodic (and therefore J is nil).

Probably the most accessible example is the ring $R_0 = M_2(GF(2))$ – i.e. the ring of 2×2 matrices over $GF(2)$. It is readily verified that $R = R_0$ has the property that

$$\text{for each} \quad x \in R\,, \quad \text{there exists} \quad ((m,n)) \quad \text{such that} \quad x^n - x^m \in N\,. \qquad (5.1)$$

Of course $J(R_0) = \{0\}$. For an example with $J \neq \{0\}$, we may take the ring of 2×2 upper-triangular matrices over $GF(2)$.

Consider the following example, discussed by Corbas in [6]. Let $R_1 = GF(2^n)\times GF(2^n)$, $n > 1$, with addition being componentwise and multiplication defined by $(a,b)(c,d) = (ac, ad + b\phi(c))$, where ϕ is a non-identity automorphism of $GF(2^n)$. Clearly, R_1 is semiperiodic with multiplicative identity $(1,0)$, $J(R_1) = N(R_1) = \{(0,b) \mid b \in GF(2^n)\}$, and $x^{2^n} - x \in N$ for all $x \in R_1$.

It is easy to see that the direct sum of two rings satisfying (5.1) is semiperiodic. Thus $R_2 = R_0 \oplus R_1$ is a semiperiodic ring with 1 for which $J(R_2)$ is a proper subset of $N(R_2)$ and $J(R_2) \nsubseteq Z(R_2)$. Let S be an algebra over $GF(2)$ obtained by adjoining an identity to a commutative nil algebra, and let $R_3 = R_0 \oplus S$. Then R_3 is semiperiodic with 1, $J(R_3)$ is a proper subset of $N(R_3)$, and $J(R_3) \subseteq Z(R_3)$.

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